FEDERAL RURAL UNIVERSITY OF PERNAMBUCO

HENRIQUE PINTO DOS SANTOS ZAIDAN

MATHEMATICAL MODELS OF WARRANTY POLICIES: A GAME THEORY PERSPECTIVE

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This thesis was considered suitable for obtaining a doctoral degree in Biometrics and Applied Statistics at the Federal Rural University of Pernambuco.

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Advisor: Prof. Dr. Cláudio Tadeu Cristino (Federal Rural University of Pernambuco/ Brazil).

Co-Advisor: Prof. Dr. Maryam Esmaeili (Alzahra University/Iran).

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This work is dedicated to my grandmother, in memoriam, who was the best woman in this world and gave me the most important lesson about life, my mother who has provided love and support, and God, the most important, who has strengthened me in my darkest hours.

Thank you so much, Bruno, Alexandre, Thyago, Salazar, Ricardo, Dov, Danilo, Eugen, Sergio, Fábio, Angelika (and her family), Deni, Polina (wife?), Jean, Yuri (brother), Diego, and Ricardo. Your friendship is a blessing.

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Finally, as in Jacob's life, I would not be able to achieve success in this academic journey if it were not for God's help.

"O Lord GOD, please remember me. Strengthen me, O God, just once more, so that with one vengeful blow I may pay back the Philistines for my two eyes."

Samson's final prayer (Judges 16:28)

"… when it starts going to your head, and you start thinking that you're a god, it's stupid because in my book there's always somebody better."

Dave Mustaine, Megadeth frontman

RESUMO

Esta tese apresenta dois modelos matemáticos de precificação de políticas de garantia formulados via teoria dos jogos. O primeiro modelo é um problema de decisão que envolve dois tomadores de decisão. O consumidor terceiriza as ações de manutenção para um agente de manutenção que oferece quatro opções de garantia com diferentes tipos de cobertura. Estratégias de equilíbrio para cada parte são dadas via equilíbrio de Nash perfeito em subjogo. O segundo modelo representa um problema de decisão com três partes visto que o fabricante do equipamento é incorporado na modelagem. O fabricante define o preço de venda do produto (incluindo os custos da garantia base), enquanto o agente de manutenção realiza as ações de reparo após o período da garantia base. Estratégias de equilíbrio são dadas via uma combinação de jogos cooperativos e não – cooperativos. O modelo trás uma coalizão estabelecida entre o fabricante e o agente, e o valor de Shapley é usado para distribuir o ganho dos jogadores e definir o preço de equilíbrio. Em ambos os modelos, nós fazemos uma análise de sensibilidade com os parâmetros do modelo e aplicamos simulação computacional para estimar a média dos custos de algumas políticas de garantia.

Palavras-chave: Políticas de garantia. Precificação. Manutenção. Teoria dos jogos. Estratégias de equilíbrio.

ABSTRACT

This thesis presents two mathematical models of pricing for warranty policies considering a game theory approach. The first model is a two-person game. The consumer outsources maintenance actions to the agent (maintenance agent), who shows four warranty options with different coverage characteristics. Equilibrium strategies for each decision-maker are obtained via the subgame-perfect Nash equilibrium. In the second model, we add the manufacturer in the modeling. Thus, a three-person game is formulated. The manufacturer defines the sale price of the product (including the base warranty costs), whereas the agent prices maintenance services. Equilibrium strategies are given through a combination of cooperative and non-cooperative solutions. The model brings a coalition between the manufacturer and the agent, and the Shapley value redistributes the payoffs and sets up the equilibrium prices. In both models, we perform a sensitivity analysis with the model parameters and apply computer simulation to estimate the expected value of warranty costs for some warranty policies.

Keywords: Warranty policies. Pricing. Product maintenance. Game theory. Equilibrium strategies.

CONTENTS

1 INTRODUCTION

This thesis discusses the pricing of warranty policies through the game theory perspective. According to Thomas and Rao (THOMAS; RAO, 1999), a warranty policy is defined mainly by two elements: 1 - the period of coverage (or warranty period), and 2 - the terms of compensation to the consumer (owner of the product) whether a failure happens. Hence, if a product is defective and not performing satisfactorily, the warranty ensures that the faulty item is either repaired or replaced by a new and non-defective item either at a reasonable cost or often at no cost to the consumer (BLISCHKE; MURTHY, 1992).

When the warranty period is standard, the compensation criterion becomes the pertinent characteristic. Generally speaking, there are two types of compensation: a free replacement warranty policy and a pro-rata warranty policy (LAM; KWOK WAI LAM, 2001).

The free replacement warranty policy bases itself on the fact that the product will be repaired (or replaced) at no cost to the consumer shall it fail before the expiration of the coverage period. Many vehicles and home appliance manufacturers have adopted this policy (JIN, 2019).

The pro-rata warranty policy bases itself on that if a product fails before the end of the warranty period, the warrantor (responsible for proving the warranty) and the consumer share the repair or replacement cost based on some product age-dependent formula. This kind of policy applies to items significantly influenced by aging, such as tires (THOMAS; RAO, 1999).

Furthermore, also there is a combined policy that involves features of both warranty policies. The usual combination begins with the free replacement policy until some specified time, and subsequently, pro-rata starts up to the end of the coverage period. An example is automobile batteries (HILL; BLISCHKE, 1987).

Each warranty policy mentioned above can be subdivided into two subgroups based on its dimension. By definition, dimension is the number of variables specified in defining the warranty limits (BLISCHKE; MURTHY, 1992). Hence, we have one-dimensional (1D) and two-dimensional (2D) policies. Specifically, 1D is characterized by an interval with a single variable (age or usage). On the other hand, 2D considers both. Below, empirical examples of both cases are presented:

- A 1D warranty policy is visible in the video game industry. For instance, Sony[1] defines the coverage for PlayStation by considering only the warranty period, which is one year from the purchase date.
- A 2D warranty policy is evident in the automobile industry. For instance, Hyundai[2] provides a warranty plan involving two variables: time and usage. Hence, its policy encompasses a warranty period of 5 years or 60,000 miles, whichever occurs first.

1.1 Game theory as a framework for modeling warranty problems

Game theory is a branch of applied mathematics that studies multi-person decision problems (GIBBONS, 1992; PETROSYAN; ZENKEVICH, 2016) whose its triad consists of the following elements; 1) decision-making, 2) conflict and 3) the optimality of a decision (VOROB'EV, 1994):

- Decision-making represents a multilateral environment. Thus, a whole set of decision-makers (players) take their strategies (decisions) from a set of all admissible decisions.
- Conflict encompasses rational questions about who participates, what the outcomes (payoffs) will be, who is interested in them, and in what way. It establishes the idea of a coalition, i.e., the complexity and degree of structure of the group of participates who make the decisions. Other questions addressed are which decisions can be made by each group, players' interests, the objective of each group (or player).
- The optimality is a set of outcomes that are declared be optimal, based on a rule of choosing (for instance, the Nash equilibrium). It may be a

[1] More details about Sony's warranty for PlayStation can be seen at https://www.playstation.com/en-us/support/warranties/ps/.

[2] More details about Hyundai's warranty information can be seen at https://m.hyundaiusa.com/assurance/2020_Owners_Handbook_Warranty_r1.pdf.

multivalued solution (more than one optimal outcome) and non-uniqueness (more than one solution).

Game theory also distinguishes between cases in which a decision-maker acts independently from all others and those who can form a group or coalition, engaging in a binding agreement that yields a certain amount of profit (FUJIWARA-GREVE, 2015; MASCHLER; SOLAN; ZAMIR, 2013).

The cooperative approach admits the players' joint actions, enabling redistribution of payoffs based on a rule, such as the Shapley value (PETROSYAN; ZENKEVICH, 2016). On the other hand, if each participant aims to achieve the maximum possible individual gain, we have the noncooperative approach (VOROB'EV, 1977).

1.1.1 Game theory characterization of warranty problems

The game theory formulation for warranty problems arises from the moment that there are two or more parties, payoff functions, and decision variables:

- Parties – consumers (equipment owners, business, and government agencies), manufacturers or retailers (responsible for manufacturing and selling the good, including maintenance services), agents (incumbent on providing the maintenance service). Each party may consist of one or more players with homogeneous or heterogeneous characteristics.
- Payoff functions – profit and utility functions, expected cost, revenue, or sales.
- Decision variables – define the equipment sale price, the coverage period, and pricing the warranty and maintenance costs. Select who will perform maintenance; when carrying out the repair actions; replace versus repair decisions.

The combination of the elements mentioned above can lead to different scenarios in the product warranty environment.

1.2 Definitions

In order to enlighten the reader, some concepts related to maintenance, after-sales services, and reliability are explained.

1.2.1 Product maintenance

Two common types of maintenance are preventive and corrective. The first one aims to control the product's degradation rate through the planned activities, such as inspection, calibration, adjustment, replacement of worn components, and replacement of degraded material. On the other hand, corrective maintenance restores a failed unit to the working state (MURTHY; KURVINEN; TÖYRYLÄ, 2016).

1.2.2 Stochastic model for the two-state characterization of degradation

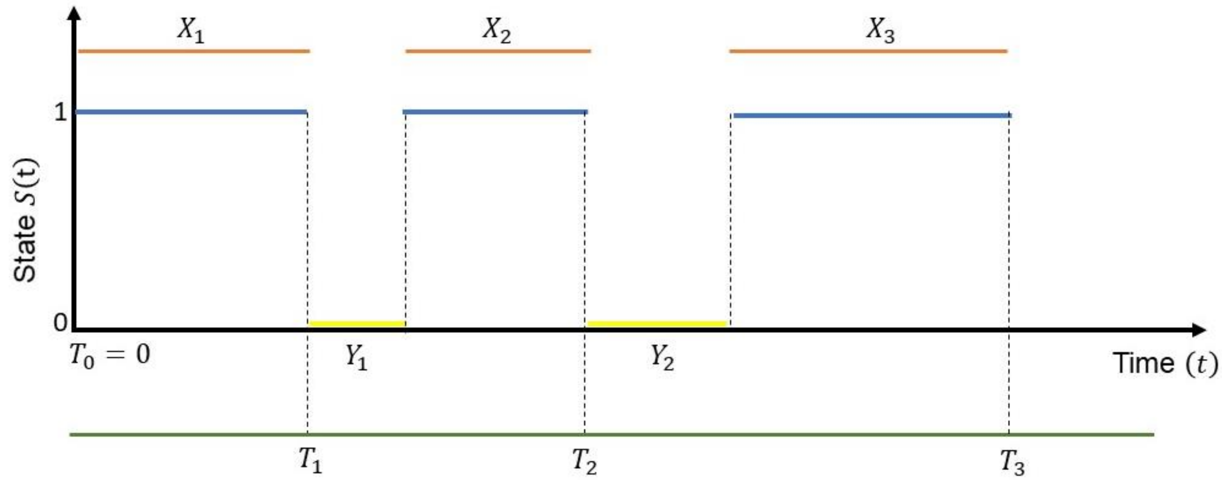
The degradation of a repairable system can be described in terms of its performance, characterized by variable $S(t)$, which indicates the product's state as a function of age (BEN-DAYA; KUMAR; MURTHY, 2016). Under a two-state characterization of degradation of $S(t)$, we have:

- $S(t) = 1$ corresponding to the product being at the working state (operational or satisfactory performance) at time t.
- $S(t) = 0$ corresponding to the product being at the failed state (unsatisfactory or not acceptable) at time t.

Figure 1 describes the failure-repair-failure cycle of the product. It starts in the working state, and after a while, it switches to the failed state. At this state, the item receives maintenance actions to return to the operational state. Consequently, there are three random variables; $X_1, X_2, \ldots$ represent times between failures, $T_1, T_2, \ldots$ are the product's ages at their respective claims, and Y_1, Y_2, ... represent repair times.

Through the interaction of these random variables, it is possible to calculate some reliability-related performance measures, such as mean time to repair the product, mean time between failures, interval availability, the expected number of product failures, and so on.

Figure 1 – Failure-repair-failure cycle considering the two-state characterization



Source: This research

1.2.2.1 Poisson process

Through the failure-repair-failure cycle of the product, it is possible to analyze the number of product failures as a counting process. Along with many types of stochastic processes (ROSS, 2010), this work focus on the Poisson process (or homogeneous Poisson process).

In broad words, the Poisson process is a discrete stochastic process in continuous time whose times between failures (interarrival times) are independent and identically distributed according to the exponential distribution. Moreover, the number of product failures follows a Poisson distribution. This stochastic process holds three properties (GNEDENKO; USHAKOV, 1995):

- Stationarity – It represents that the distribution function of the number of failures for any given interval depends only on the length of that interval and not on its position in time. In particular, the probability of occurrence of k failures during the interval of time from τ to $\tau + t$ does not depend on τ and is a function only of t and k.
- Memorylessness – It represents the probability of occurrence of k failures during the time interval $(t, t + \tau)$ does not depend on knowledge of how many failures have occurred earlier or how they occurred. As a result, the previous history does not affect the density function of the number of failures.

- Ordinarity – This property implies that the probability of an appearance of more than one failure in an infinitesimally small interval Δt goes to 0.

Finally, we mention that when the Poisson process does not have the stationary increment property is called the non-homogeneous Poisson process. Under this stochastic process, the failure rate varies as a function of time (GALLAGER, 2013).

1.2.3 After-sales services

Table 1 summarizes three types of after-sales services are related to product maintenance commonly applied to the post-sales services market.

Table 1 – Types of after-sales services

Types	Description
Base Warranty	A base warranty (or warranty) is a contract between the consumer, the product owner, and the original equipment manufacturer (OEM or manufacturer) that starts after the product sale. It requires the OEM to repair, replace, or provide compensation to the consumer in the case of product failure along with the coverage period. Its cost is factored into the product's price.
Extended Warranty	The extended warranty is an optional service that provides additional coverage for the product after the base warranty. The consumer pays a premium to acquire it. Usually, the manufacturer offers it when the consumer purchases the product. Retailers, insurance companies, and other parties may provide it as well. This service covers only corrective maintenance costs.
Maintenance Contracts	A maintenance contract is similar to the concept of an extended warranty, a non-mandatory coverage. The maintenance agent (or agent) carries out the maintenance for an agreed period via preventive (and/or corrective) maintenance actions.

Source: (MURTHY; JACK, 2014; RAHMAN; CHATTOPADHYAY, 2015)

1.3 Motivation

I have been studying warranty policies since 2014. At that time, I was a master's student in production engineering (management engineering) and did not have the experience to do research. In the beginning, I did not enjoy this area. If I could have changed my study field, I would preferably work on forecasting methods or something related to economics science. When I began researching this topic, some unexpected family problems happened. They were related to my grandmother's health, which upset me. I was afraid. She had a stroke on December 30th, 2014. Furthermore, I did not like the posture of my former advisor concerning the progress of my master's thesis. I desired to put him off and seek a new professor.

The last problem was that all theoretical background associated with this topic was written in English, and, at that period, I did not know the language. It was an arduous period to start researching without academic support.

Amid these problems, I began to read the paper called A Stochastic Model for Service Contract (MURTHY; ASGHARIZADEH, 1998), which has become my academic afflatus.

I absolutely can say that this paper changed my perspective about the warranty study. It is a multi-person decision problem related to maintenance outsourcing between a consumer (the product owner) and a maintenance agent who offers two maintenance options: a maintenance contract and a service on demand (no service contract). The product is a repairable system, and the failures and repair times follow an exponential distribution. The model is developed under a game theory approach, yielding equilibrium strategies for all decision-makers via the Stackelberg solution. I was fascinated by how the authors developed the structure of the maintenance options and found an equilibrium through the game theory approach.

Furthermore, I sent an e-mail to one of the authors, Pra Murthy, who was attentive to me. He provided me additional information about this topic and gave me materials that improved my research. In possession of all this academic material, I noticed that I could contribute to this field.

From a market perspective, modern manufacturing presents rapidly changing technologies, fierce competition among companies, besides often nearly identical products (MURTHY; DJAMALUDIN, 2002). Hence, the product-service system has gained prominence, enabling fill the consumer needs and drive up the profitability of manufacturers and maintenance agents by selling extra services such as product maintenance and repair services (MONT, 2002).

One of the significant motivations for companies to buy maintenance services and extended warranties is to save maintenance costs. Through maintenance outsourcing, the companies outsource product maintenance, which can help them reduce the operation costs, labor, and spare parts inventory expenses (HAMIDI, 2016). According to Campbell (CAMPBELL, 1995), 35% of North American businesses have considered outsourcing as a portion of their maintenance needs.

From management's point of view, express warranties have two core objectives - promotion and protection. The promotional purpose is to encourage purchases by reducing risks for the consumer. Meanwhile, the protection reason is to guard the seller (manufacturer or retailer) from unreasonable purchase claims (UDELL; ANDERSON, 1968).

According to Emons (EMONS, 1989), the existence of the warranty is associated with three motives; i) insurance, ii) signaling, and iii) incentive. The insurance motive assumes that the consumers are more risk-averse than sellers. This fact implies that the insurance works to reduce the impact against the event of product failure.

The signaling motive is that sellers use the warranty as a qualitative attribute signal for selling their products. As a result, more reliable products imply lower costs and extensive coverage. The incentive motive views the warranty as a device for firms not to cheat on the product quality. Furthermore, the warranty as a marketing strategy can extract the consumer surplus to the seller, increasing their profit.

On the other hand, the provision of any warranty policy involves costs, called warranty servicing costs (SHAFIEE; CHUKOVA, 2013). They are the costs of rectifying a faulty item during the warranty period. These include repair, labor, parts, administration, handling costs, replacement costs, and others (RAHMAN, 2007). According to Murthy (MURTHY, 2007), warranty costs represent a fraction of the sale price, varying from 1% - 10% depending on the product and the manufacturer. Such costs also are random variables since claims under the warranty period are uncertain because the failure is a stochastic event (RAHMAN; CHATTOPADHYAY, 2015).

By numbers, in the world aviation sector in 2018, the world's ten biggest civilian aircraft manufacturers' spent over $1 billion on warranty claims (WARRANTY WEEK, 2019). Apple's annual report in 2019 indicated that the warranty and related costs were $3.57 billion, 6.46% of its net income (APPLE INC., 2019). Moreover, American consumers bought $44.7 billion for their protection plans (mobile phones, high-tech products, vehicles, consumer electronics, furniture, and PCs) in 2017 (WARRANTY WEEK, 2018).

Considering long-term warranty periods, farm equipment manufacturers set up their strategies as follows. Summers Manufacturing provides a 10-year limited warranty program on all new land roller equipment purchases. Furthermore, this warranty can be transferred to a second owner for a transfer fee (EQUIPMENT, 2021).

Therefore, studying warranty policies is very important due to the amount of money involved, the market opportunities to be explored, and the involvement of many different stockholders with distinct needs. Predicting failures, cost models, and meet consumer needs are challenges handle designing a warranty policy and its price.

Finally, as in Moses's life, I would like to say that every day of my student life has been challenging, but in the end, God's oath to Abraham and his descendants is authentic and prevails.

To do a Ph.D. without the financial aid from my postgraduate program was an incredible challenge. Without God's help, I certainly would have given up and looked for a better opportunity. Moreover, the conclusion of this step in my life represents a journey that I started in 2016 with my grandmother. Finally, I am grateful for the financial support from my mother since, without her aid, I would do another activity.

I hope to publish my thesis papers with the support of Professor Maryam Esmaeili from Alzahra University after my Ph.D. I wish to be in an academic center where I can feel appreciated to improve my academic skills.

This research is the end of a cycle I started in 2009.

1.4 Thesis statement

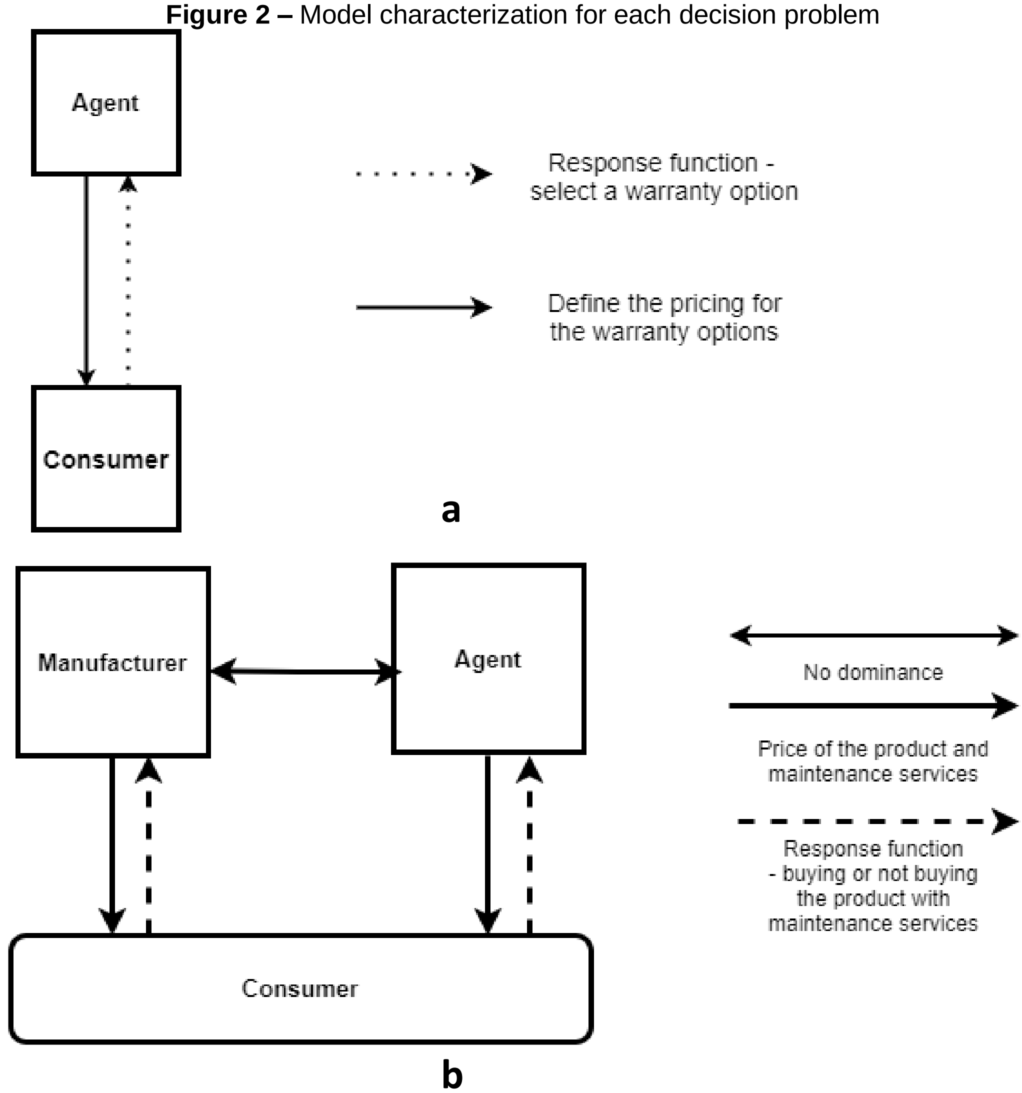
This thesis deal with the warranty pricing for a given coverage period. Thus, two theoretical models form this document. For each decision problem, we go into greater detail a complete characterization of equilibrium strategies. For some warranty policies, we use a simulation model to compute the expected warranty costs.

The first model is a sequential two-person game applied to the maintenance service contract context. The consumer outsources product maintenance to the agent who offers four warranty options. Equilibrium strategies are formulated by subgame-perfect Nash equilibrium.

The second model is a three-person game defined among a manufacturer, an agent, and a consumer. Under a mix of non-cooperative and cooperative solutions, equilibrium strategies are developed for all decision-makers. The model brings a coalition between the manufacturer and the agent that pays a sequential game with the consumer. The manufacturer defines the equipment price (including the base warranty costs), while the agent prices maintenance services. Thus, for a given pair of prices set up by the coalition, the consumer replies, buying or not buying the product with maintenance services.

Figure 2 gives a visual perspective for each model, considering each scenario of the application.

Figure 2 – Model characterization for each decision problem



Source: This research

Figure 2 (a) presents the relationship between the agent and the consumer. For pricing offered by the agent, the consumer chooses a warranty option through their payoff function.

Figure 2 (b) considers a scenario of three parties in a hierarchical system with a double subordination division. Under this context, at the upper level, the manufacturer and the agent take decisions simultaneously, influencing the consumer, the bottom level, that gives a reply through their payoff function.

1.4.1 Connection between the models

Both models rely on the same assumptions of reliability-related performance measures, consider multiple decision-makers, sequential structure (two-stage game with complete information), and deal with the pricing of the warranty for a given period. Furthermore, they use the game theory approach to find an equilibrium (subgame-perfect Nash equilibrium). The second model extends the previous one by including a new player (the manufacturer) in the modeling. As a result, the decision problem becomes more complex since the warranty pricing problem holds two parties; the manufacturer and the agent that set their prices simultaneously.

1.5 Research methodology

We confine our analysis to quantitative modeling of the warranty pricing following the systems approach proposed by Murthy and Blischke (MURTHY; BLISCHKE, 1992). It is a multistep process applied to solving real-world problems involving several stages:

- Stage 1: System characterization.
- Stage 2: Mathematical modeling.
- Stage 3: Analysis of the mathematical model.
- Stage 4: Interpretation of analysis.

System characterization represents the first stage of the systems approach. It is a process of simplification and idealization, being seen as a descriptive model that includes the relevant characteristics of the problem. Hence, parameters, variables, the set of decision-makers, and their relationships are detailed. In this part, the item failure pattern and the design of the warranty policy also are specified. The effort needed to execute Stages 2-3 depends on the complexity established on system characterization.

In Stage 2, mathematical modeling represents the transformation of the system characterization into a mathematical description, yielding a mathematical model. It conjoins the variables and relationships defined in Stage 1 to an abstract mathematical formulation, generating a one-to-one correspondence. The main advantage of mathematical models is that they allow qualitative and quantitative study using well-established mathematical techniques.

The third stage involves the mathematical tools and techniques that can be either analytical and (or) computational to find the model solution. The computational method can be applied when the analytical approach provides models that hold oversimplifications that lack realism because they fail to account for many important aspects of the actual warranty process, and (or) they quickly lead to intractable mathematics (HILL; BEALL; BLISCHKE, 1991).

Finally, Stage 4 represents the analysis of results obtained from the mathematical modeling that yields the solution. It is essential to ensure that the model used is adequate for solving the problem addressed in the first stage. Statistical methods can be applied to perform the model validation.

1.6 Research objectives

1.6.1 Main objective

This thesis develops a framework to study the pricing of warranty policies under a game theory formulation. Two quantitative models form this document. Each one has its purposes, aiming to fill a particular gap in the warranty literature:

- We proposed a sequential game, considering the agent’s and consumer’s perspectives. The consumer, the owner of a product, outsources maintenance actions to the agent that offers four warranty options with different types of coverage. Equilibrium strategies for all players are given by subgame-perfect Nash equilibrium. The consumer chooses a warranty option, and the agent prices them. This study covers the warranty maintenance costs, decision-making process, and pricing. This research bases on the work proposed by Murthy and Asgharizadeh (MURTHY; ASGHARIZADEH, 1998).

- The second model aims to fill the gap present in the warranty literature about modeling a three-person game. Only three papers consider an interaction among the manufacturer, the agent, and the consumer (ESMAEILI; SHAMSI GAMCHI; ASGHARIZADEH, 2014; GAMCHI; ESMAEILI; MONFARED, 2013; TALEIZADEH; SHERAFATI, 2019). Hence, we propose a new approach to study this problem. Our modeling is based on a hierarchical structure with a double subordination division proposed by Petrosyan and Zenkevich (PETROSYAN; ZENKEVICH, 2016). The manufacturer specifies the equipment price (including the base warranty costs), whereas the agent prices maintenance services. The consumer decides whether to buy or not the product with maintenance services. Equilibrium strategies are defined via a mix of cooperative and non-cooperative solutions. The model brings a coalition between the manufacturer and the agent, and the Shapley value redistributes the gain between the players and defines equilibrium prices.

1.6.2 Specific objectives

The following specific objectives are defined to attain the main goals of the thesis:

- To identify and characterize the key elements of game theory inserted in warranty problems related to the pricing. Thus, we explore the set of players and strategies for each decision-maker, the order of moves, characteristic functions, equilibrium strategies, and payoff functions.
- To give an accurate description of how the warranty costs are computed for each warranty policy. Furthermore, we emphasize as the simulation model can be convenient for warranty analysis.

1.7 Contributions

We hope this doctoral thesis becomes a guide for everyone who desires to study warranty policies through a game theory perspective. This research presents two problems that deal with price-warranty combinations considering multiples decision-makers under a theoretical approach. We show a real problem confronted by many manufacturers, retailers, maintenance agents, and other parties related to the pricing of the warranty for a given period, considering its costs that are random and the consumer's decision to buying it.

1.8 Thesis organization

Besides this introduction, this thesis contains three additional chapters, which are described as follows:

Chapter 2 presents the first model, a decision problem between a consumer and an agent, formulated as a sequential two-person game. We explain its formulation and the procedure to find the equilibrium strategies through the subgame-perfect Nash equilibrium. Furthermore, we perform a sensibility analysis of the model parameters and provide some extensions.

Chapter 3 presents the second model, a hierarchical game considering three players: a manufacturer, an agent, and a consumer. For this work, equilibrium strategies are formulated through a mix of cooperative and non-cooperative solutions. The model brings a coalition between the manufacturer and the agent, and the Shapley value provides the payoffs for all decision-makers and defines equilibrium prices. This chapter is a draft, can be deepened for futures works. It was developed during my second interchange at Saint Petersburg University under the supervision of Professor Leon Petrosjan.

Chapter 4 summarizes the main contributions obtained of these two models, draws general conclusions of warranty policies under a game theory formulation, and indicates suggestions for future research in this academic field.

Finally, we mention that the notation for Chapters 2 and 3 belongs to itself. Thus, they are not interchangeable.

"*I did not stay on the street long because I do not have any pretentious values in life. I had to start learning music just for the dollar sign*."

Dave Mustaine

2 FIRST ESSAY. MAINTENANCE OUTSOURCING: A NEW LOOK THROUGH GAME THEORY AND STOCHASTIC OPTIMIZATION

Nowadays, maintenance outsourcing represents a trend that many consumers have adopted with their assets. A maintenance agent (or agent) provides this service under a maintenance contract that includes terms of compensation related to the product's performance along with a coverage period. In this work, we go deeper into this problem through a model set up between an owner of a piece of equipment (consumer) and an agent who offers four warranty options with different coverage characteristics. Equilibrium strategies for each decision-maker are modeled via the subgame-perfect Nash equilibrium. The model deals with a pricing problem where the agent prices the warranty options, and the consumer selects one. Numerical examples, sensitivity analyses with the model parameters, and managerial insights are presented in this paper to demonstrate its purpose.

Keywords: Two-person game. Warranty options. Maintenance. Subgame-perfect Nash equilibrium. Pricing.

2.1 Highlights

- Propose four warranty options based on free replacement warranty and pro-rata policies taking into consideration corrective actions for a repairable product.
- Pricing of warranty options through the consumer's reservation prices to the agent obtains the maximum expected profit in equilibrium.
- A simulation model to compute the expected cost for the rebate terms associated with the warranty options.
- We provide equilibrium strategies for both decision-makers for a fixed-warranty period.

2.2 Model's motivation

The equipment owner (consumer) outsources product maintenance to a maintenance agent (agent) who shows four warranty options with different coverage mechanisms. Hence, the agent prices them, and the consumer chooses one. The main point is how to obtain equilibrium strategies for this decision problem? The consumer deals with four-dimension space that are the warranty prices defined by the agent. We explain under certain conditions how the equilibrium is reached. Another fascinating achievement is what warranty options are preferable to others.

2.3 Notation list

2.3.1 Input parameters

L: equipment's lifetime;

λ: failure rate;

μ: repair rate;

τ: threshold time to finish the maintenance without penalty;

P_E: equipment's sale price;

α: penalty cost per time;

R: revenue;

O_i: warranty option; $= 1{,}2{,}3{,}4$;

W_1: period of the coverage related to warranty option O_2;

W_2: period of the coverage related to warranty option O_3;

ξ: parameter defined by the agent related to the maintenance cost over W_1;

$N(t)$: number of product failures during period t;

C_A: agent's maintenance cost;

$\Pi_C(i,)$: consumer's payoff function under warranty option $i;\ i = 1{,}2{,}3{,}4$;

$\Pi_A(,i)$: agent's payoff function under warranty option $i;\ i = 1{,}2{,}3{,}4$;

P_A: agent's set of strategies;

X: set of the warranty options;

2.3.2 Decision variables

$\tilde{p}_A$: agent's strategy;

$x(\tilde{p}_A)$: consumer's strategy.

2.3.3 Consumer's reservation prices

$\bar{P}_M$: consumer's reservation price for warranty option O_1;

$\bar{P}_R$: consumer's reservation price for warranty option O_2;

$\overline{PW}_2$: consumer's reservation price for warranty option O_3;

$\bar{P}_T$: consumer's reservation price for warranty option O_4.

2.3.4 Equilibrium strategies

$\dot{p}_A$: agent's equilibrium strategy;

$\dot{x}(\dot{p}_A)$: consumer's equilibrium strategy.

2.4 Model formulation

This section illustrates the decision problem related to the post-purchase decision between the equipment owner and the agent who carries out maintenance.

The consumer outsources the maintenance of a piece of equipment (repairable system) to the agent, who offers four warranty options with different coverage features. The length of the warranty is the product's lifecycle (L). The good generates revenue R per unit time in the working state. In contrast, in the failed state, the item receives maintenance actions from the agent and does not generate revenue for the consumer.

2.4.1 Equipment failures and repairs

The time between two consecutive failures follows the exponential distribution whose density function is given by

$$f(t;\lambda) = \lambda e^{-\lambda t},$$

where $\lambda(> 0)$ is the parameter and $t(\geq 0)$ is the time. Under this probability distribution, the failure rate is λ, indicating that the current or future reliability properties of the product do not change with time and, consequently, do not depend on the amount of operating time since the moment of switching the equipment on, the memoryless property (GNEDENKO; USHAKOV, 1995). This assumption is valid when the failure rate is in the second region of the bathtub curve (BEN-DAYA; KUMAR; MURTHY, 2016). Furthermore, as the failure rate is constant, the repair brings the equipment to a like-new condition (perfect repair) (PULCINI, 2003).

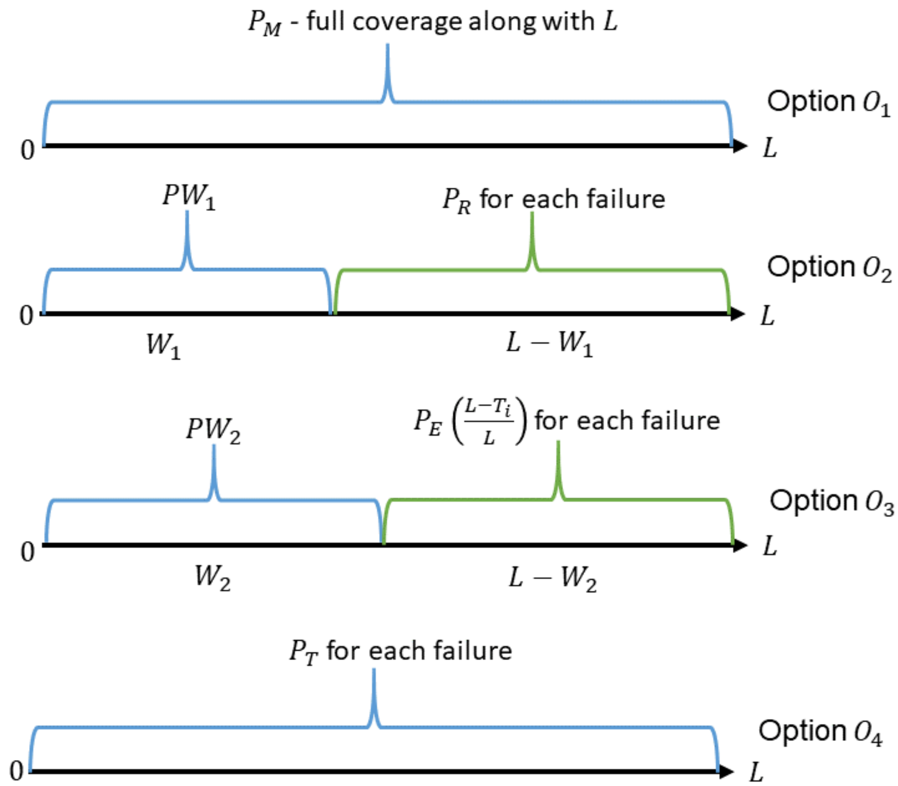
As the times between failures follow an exponentially distributed, the agent only carries out corrective maintenance (MURTHY; ASGHARIZADEH, 1998). Y_i is the agent's repair time at the ith failure that also follows an exponential distribution whose $\mu(> 0)$ represents the repair rate.

2.4.2 Warranty options

- Option O_1− Full maintenance contract. Under this option, for a fixed price P_M, the agent repairs the product failures over $[0, L)$ at no additional cost. If equipment is not repaired within a time τ after the claim, the agent incurs a penalty. The penalty structure is $\alpha(Y_i - \tau)$ if $Y_i > \tau$ and zero otherwise. α is the penalty cost per time, expressed in monetary units.
- Option O_2− Partial maintenance contract. Under this option, for a fixed price PW_1, the agent agrees to repair the product along with the coverage period of length $W_1(< L)$. After W_1, the agent charges a price P_R for each repair intervention until the end of the product's lifecycle. There is no penalty regarding the time taken to rectify the failure.
- Option O_3− Hybrid maintenance contract. Under this option, for a fixed price PW_2, the agent agrees to repair the product along with the coverage period of length $W_2(< L)$. After W_2, the maintenance price is a linear function of the time service of product, considering the equipment's sale price (P_E). Hence, the consumer pays $P_E\left(\frac{L-T_i}{L}\right)$ for each repair intervention. T_i is the age of the item at the ith failure. There is no penalty regarding the time taken to rectify the failure.
- Option O_4− Services on demand. Under this option, the agent charges a price P_T for each repair intervention over $[0, L)$. There is no penalty regarding the time taken to rectify the failure.

Figure 3 summarizes the warranty options provided by the agent, considering its price and coverage period.

Figure 3 – Design of the warranty options provided by the agent



Source: This research

2.4.3 Agent's decision problem

For the agent, the revenue under Option O_1 is constant, being P_M. For other options, it is a random variable indexed with the number of product failures. Furthermore, the warranty costs are stochastic since they also are affected by product failures. In Option O_1, there is an additional cost related to the penalty time. In summary, for all cases, the agent's profit is stochastic.

As W_1, W_2, and L are given, then the agent prices the warranty options. PW_1 from Option O_2 follows the same pattern seen in Esmaeili *et al.* (ESMAEILI; SHAMSI GAMCHI; ASGHARIZADEH, 2014). It represents a linear function of the expected number of product failures over the coverage period of length W_1 according to

$$PW_1 = \xi \mathrm{E}[N(W_1)], \tag{2.1}$$

where $\xi(> 0)$ is a parameter defined by the agent considering the maintenance cost and $N(W_1)$ is the number of product failures over the coverage period of length W_1. In practice, PW_1 should be upper than the agent's warranty cost total along with W_1. Consequently, in option O_2, the agent only sets up P_R.

Let P_A denote the agent's set of strategies and $\tilde{p}_A$ is a strategy from this set, where

$$\tilde{p}_A = (P_M, P_R, PW_2, P_T),$$
$$\tilde{p}_A \in P_A.$$

Note that $\tilde{p}_A$ is a vector of prices associated with the warranty options.

2.4.4 Consumer's decision problem

The consumer makes their decision considering the prices defined by the agent. In Option O_1, the consumer incurs a fixed cost and receives financial compensation for the downtime in repairing the device. On the other hand, considering other options, their warranty cost is stochastic, associated with product failures. Besides, note that in Option O_3, after W_2, the consumer's warranty cost decreases for each claim.

Let X be the set of warranty options, and $x(\tilde{p}_A)$ is the consumer's strategy, where

$$X = (O_1, O_2, O_3, O_4),$$
$$x(\tilde{p}_A) \in X.$$

Table 2 describes the possible values of the consumer's strategy.

Table 2 – Consumer's decision strategy

$x(\tilde{p}_A)$	**Description**
1	The consumer chooses Option O_1.
2	The consumer chooses Option O_2.
3	The consumer chooses Option O_3.
4	The consumer chooses Option O_4.

Source: The author

2.4.5 Assumptions

1. It is assumed the equipment owner must choose a warranty option provided by the agent. Thus, the consumer does not abandon the good after a failure.
2. All elements of the structure of the game are known (complete information).

3. The agent's repair times are infinitesimal concerning mean times between failures, $\mu^{-1} \ll \lambda^{-1}$. Consequently, the number of product failures follows a Poisson distribution and the revenue generated by the equipment over its useful life can be approximated by RL.
4. The agent’s warranty cost per failure does not change over time, corresponding C_A per failure.

Assumption 2 implies that the payoff functions are identified for both players, i.e., the reliability-related performance measures of the product are known to the decision-makers.

Assumption 3 states that the sum of repair times is negligible compared to the product's useful life. Thus, even with the presence of downtimes, they do not affect the product's availability. The failure-repair-failure cycle of the product follows the homogeneous Poisson process (LAD; SHRIVASTAVA; KULKARNI, 2016).

Assumption 4 may be supported if the labor and diagnosis costs dominate the warranty servicing costs (GLICKMAN; BERGER, 1976).

2.4.6 Payoff functions

The players' payoff functions are profit functions that incorporate randomness due to reliability-related performance measures. They are expressed in terms of the expected value of the monetary outcome.

2.4.6.1 Consumer’s expected gain

The consumer’s expected gain under option O_1 is given by

$$\mathrm{E}[\Pi_C(1,\tilde{p}_A)] = RL + \alpha \mathrm{E}_{N(L)}\left[\mathrm{E}_{Y_i-\tau}\left[\sum_{i=0}^{N(L)} max(0, Y_i - \tau)\,|N(L)\right]\right] - P_M, \qquad (2.2)$$

where $N(L)$ is the number of product’s failures over $[0, L)$.

The expression $\alpha \mathrm{E}_{N(L)}\left[\mathrm{E}_{Y_i-\tau}\left[\sum_{i=0}^{N(L)} max(0, Y_i - \tau)\,|N(L)\right]\right]$ represents the expected value of the financial compensation to be received by the consumer related to the penalty time. Formally, this double expectation is the expected value of the penalty time (a truncated Erlang random variable) by conditioning it on the number of product failures (a Poisson random variable). It formulates under the concept of the law of total expectation from conditional expectation (ROSS, 2010), i.e., the number of product failures affects the penalty time. Thus, there is a dependency relationship between these two random variables. $N(L)$ also influences the average penalty time.

Next, the consumer's expected gain under option O_2 is given by

$$\mathrm{E}[\Pi_C(2,\tilde{p}_A)] = RL - PW_1 - P_R \mathrm{E}[N(L - W_1)], \quad (2.3)$$

where $N(L - W_1)$ is the number of product's failures along with the coverage period of length $[W_1, L)$.

Then, the consumer's expected gain under option O_3 is given by

$$\mathrm{E}[\Pi_C(3,\tilde{p}_A)] = RL - PW_2 - P_E \mathrm{E}_{N(L-W_2)}\left[\mathrm{E}_{T_i}\left[\left(\sum_{i=0}^{N(L-W_2)} \frac{T_i}{L}\right) - N(L - W_2) | N(L - W_2)\right]\right], \quad (2.4)$$

with $T_0 = 0$. Where $N(L - W_2)$ is the number of product's failures along with the coverage period of length $[W_2, L)$.

The expression $P_E \mathrm{E}_{N(L-W_2)}\left[\mathrm{E}_{T_i}\left[\left(\sum_{i=0}^{N(L-W_2)} \frac{T_i}{L}\right) - N(L - W_2) | N(L - W_2)\right]\right]$ represents the consumer's expected warranty cost after W_2. It follows the same idea of the expected value of the penalty time in option O_1 $(\mathrm{Eq.}\, 2.2)$, i.e., holds the concept of the law of total expectation. As a result, this expected value depends on the summation failure times (an Erlang random variable) and the number of failures after W_2 (a Poisson random variable), existing a dependency relationship between these variables. Formally, this double expectation means the expected value of the summation of the failure times by conditioning it on the number of product failures after W_2.

It is worth mentioning that the failure times are a sequence of increasing random variables $(T_i < T_{i+1})$. T_i starts after W_2 and multiple failures cannot occur simultaneously.

Finally, the consumer's expected gain under option O_4 is given by

$$\mathrm{E}[\Pi_C(4,\tilde{p}_A)] = RL - P_T \mathrm{E}[N(L)]. \quad (2.5)$$

2.4.6.2 Agent's expected profit

The agent's expected profit under option O_1 is given by

$$\mathrm{E}[\Pi_A(\tilde{p}_A, 1)] = P_M - \alpha \mathrm{E}_{N(L)}\left[\mathrm{E}_{Y_i-\tau}\left[\sum_{i=0}^{N(L)} max(0, Y_i - \tau)\,|N(L)\right]\right] - C_A \mathrm{E}[N(L)]. \quad (2.6)$$

Next, the agent's expected profit under option O_2 is given by

$$\mathrm{E}[\Pi_A(\tilde{p}_A, 2)] = PW_1 + P_R \mathrm{E}[N(L - W_1)] - C_A \mathrm{E}[N(L)]. \quad (2.7)$$

Then, the agent's expected profit under option O_3 is given by

$$\mathrm{E}[\Pi_A(\tilde{p}_A, 3)] = PW_2 + P_E \mathrm{E}_{N(L-W_2)}\left[\mathrm{E}_{T_i}\left[\left(\sum_{i=0}^{N(L-W_2)} \frac{T_i}{L}\right) - N(L - W_2)|N(L - W_2)\right]\right] - C_A \mathrm{E}[N(L)]. \quad (2.8)$$

Finally, the agent's expected profit under option O_4 is given by

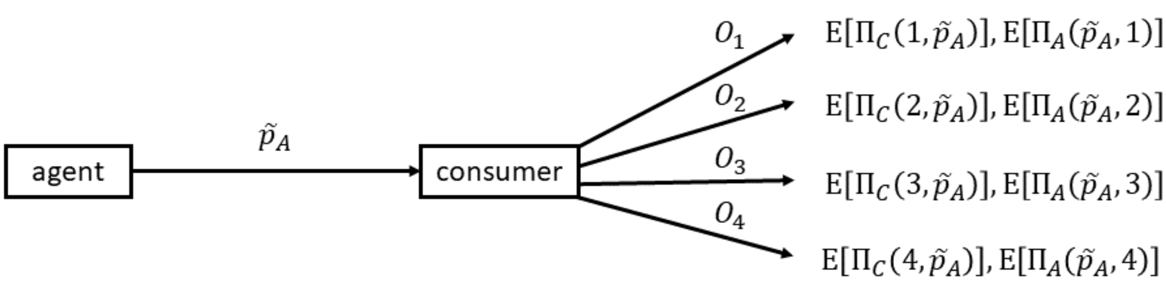
$$\mathrm{E}[\Pi_A(\tilde{p}_A, 4)] = (P_T - C_A)\mathrm{E}[N(L)]. \quad (2.9)$$

$N(L)$, $N(W_1)$, $N(L - W_1)$, and $N(L - W_2)$ are Poisson distributed with means λL, λW_1, $\lambda(L - W_1)$, and $\lambda(L - W_2)$, respectively. In a homogeneous Poisson process, the expected number of failures is proportional to time length due to the assumption of stationary increments (GNEDENKO; USHAKOV, 1995).

2.5 Model solution

The decision problem is a sequential game divided into two stages. At first, the agent prices the warranty options. Next, the consumer chooses only one, and the payoffs are computed. Figure 4 provides the extensive form representation of the model.

Figure 4 – Game tree of the model



Source: This research

Since the model presents complete information and finite horizon, equilibrium strategies are calculated through the subgame-perfect Nash equilibrium (MAZALOV, 2014). For a given $\tilde{p}_A$, the consumer compares their expected payoff for each warranty option and selects the option that obtains the highest gain, corresponding to their equilibrium strategy - $\dot{x}(\dot{p}_A)$. Based on that, the agent anticipates $\dot{x}(\dot{p}_A)$ and sets up their price to earn the maximum expected profit, corresponding to their equilibrium strategy - $\dot{p}_A$. Formally, we can say that the agent learns the consumer's set of best responses to any $\tilde{p}_A$, and by having this information, maximizes their payoff by selecting $\dot{p}_A$.

2.5.1 Set of feasible solutions

2.5.1.1 Consumer's point of view

The consumer seeks the warranty option that maximizes their expected payoff. If two or more options provide the same expected gain, the consumer is indifferent among them.

The consumer's reservation price (VARIAN, 1992), computed through the payoff function of the consumer $(\text{Eq. }2.2, \text{Eq. }2.3, \text{Eq. }2.4, \text{Eq. }2.5)$, represents the maximum price the consumer is willing to pay for each warranty option. Above it, the consumer does not select the warranty option once their expected payoff becomes negative.

From $\text{Eq. }2.2$, the consumer's reservation price for option O_1 is

$$\bar{P}_M = RL + \alpha \mathrm{E}_{N(L)}\left[\mathrm{E}_{Y_i-\tau}\left[\sum_{i=0}^{N(L)} max(0, Y_i - \tau)\,|N(L)\right]\right], \qquad (2.10)$$

where $\bar{P}_M$ represents the consumer's reservation price for option O_1.

From $\text{Eq. }2.1, \text{Eq. }2.3$, the consumer's reservation price for option O_2 is

$$\overline{\mathrm{P}}_R = \frac{RL - \xi\lambda W_1}{\lambda(L - W_1)}, \qquad (2.11)$$

where $\bar{P}_R$ represents the consumer's reservation price for option O_2.

From $\text{Eq. }2.4$, the consumer's reservation price for option O_3 is

$$\overline{\mathrm{PW}}_2 = RL - P_E \mathrm{E}_{N(L-W_2)}\left[\mathrm{E}_{T_i}\left[\left(\sum_{i=0}^{N(L-W_2)} \frac{T_i}{L}\right) - N(L - W_2)|N(L - W_2)\right]\right], \qquad (2.12)$$

where $\overline{\mathrm{PW}}_2$ represents the consumer's reservation price for option O_3.

From $\text{Eq. }2.5$, the consumer's reservation price for option O_4 is

$$\overline{\mathrm{P}}_T = \frac{R}{\lambda}, \qquad (2.13)$$

where $\bar{P}_T$ represents the consumer's reservation price for option O_4.

2.5.1.2 Agent's point of view

The agent has a range of possibilities to define the pricing of the warranty options. At the bottom, the warranty prices must be equal to the warranty costs. If the agent considers the consumer's reservation price to construe their pricing, their expected profit is the highest. On the other hand, the expected gain of the consumer (and consumer surplus) is zero. Under this behavior, the agent extracts the maximum possible amount from the consumer, such as a monopolist applies first-degree price discrimination (VARIAN, 1989).

2.5.2 Strategy profile of equilibrium

The strategy profile of equilibrium can be achieved using consumer's reservation prices. The agent enforces their strategy by using such prices, manipulating the equilibrium strategy of the consumer. If the agent requires the consumer to select a specific warranty option, the other options are priced above the consumer's reservation prices. Consequently, the product owner chooses the option wished by the agent. The equilibrium path is organized as follow:

- If the agent enforces that the consumer chooses Option O_1, the strategy profile of equilibrium is $\mathrm{E}[\Pi_A(\dot{p}_A, 1)] = P_{MC} - \alpha \mathrm{E}_{N(L)}\left[\mathrm{E}_{Y_i - \tau}\left[\sum_{i=0}^{N(L)} max(0, Y_i - \tau)\,|N(L)\right]\right] - C_A \mathrm{E}[N(L)]$, $\dot{p}_A = (P_M = \bar{P}_M, P_R > \bar{P}_R, PW_2 > \overline{PW_2},\ P_T > \bar{P}_T)$, and $\dot{x}(\dot{p}_A) = 1$.
- If the agent enforces that the consumer chooses Option O_2, the strategy profile of equilibrium is $\mathrm{E}[\Pi_A(\dot{p}_A, 2)] = PW_1 + P_R \mathrm{E}[N(L - W_1)] - C_A \mathrm{E}[N(L)]$, $\dot{p}_A = (P_M > \bar{P}_M, P_R = \bar{P}_R, PW_2 > \overline{PW_2},\ P_T > \bar{P}_T)$, and $\dot{x}(\dot{p}_A) = 2$.
- If the agent enforces that the consumer chooses Option O_3, the strategy profile of equilibrium is $\mathrm{E}[\Pi_A(\dot{p}_A, 3)] = PW_2 + P_E \mathrm{E}_{N(L-W_2)}\left[\mathrm{E}_{T_i}\left[\left(\sum_{i=0}^{N(L-W_2)} \frac{T_i}{L}\right) - N(L - W_2)|N(L - W_2)\right]\right] - C_A \mathrm{E}[N(L)]$, $\dot{p}_A = (P_M > \bar{P}_M, P_R > \bar{P}_R, PW_2 = \overline{PW_2},\ P_T > \bar{P}_T)$, and $\dot{x}(\dot{p}_A) = 3$.

- If the agent enforces that the consumer chooses Option O_4, the strategy profile of equilibrium is $\mathrm{E}[\Pi_A(\dot{p}_A,4)] = (P_T - C_A)\mathrm{E}[N(L)]$, $\dot{p}_A = (P_M > \bar{P}_M, P_R > \bar{P}_R, PW_2 > \overline{PW_2}, P_T = \bar{P}_T)$, and $\dot{x}(\dot{p}_A) = 4$.

Finally, if the set of prices are equal to the consumer's reservation prices, the consumer is indifferent among the warranty options since their expected gain is zero for each option.

2.5.3 Computational approach

This section presents computational simulation as an alternative to estimate the penalty time expected incurred by the agent (and, consequently, the consumer's financial compensation) in option O_1 and the consumer's warranty expected cost after W_2 related to option O_3.

The computational approach applies when the warranty cost models deal with intractable mathematics (HILL; BEALL; BLISCHKE, 1991). Hence, to overcome this problem, a simulation model estimates the mean of these random variables that affect the players' payoffs.

2.5.3.1 Penalty time - Option O_1

This simulation develops an empirical estimator to estimate the mean of the penalty time incurred by the agent.

The statistical procedure divides into two steps. First, to identify the value of $N(L)$. Based on that, generate repair times (Y_i , $i = 1,2,\dots,N(L)$) and compare them with the threshold time to finish the repair without penalty (τ).

Under the third assumption, the number of product failures follows a Poisson distribution, and the failure-repair-failure cycle of the product is given by the homogeneous Poisson process. Thus, $N(L)$ is simulated according to the procedure seen in Ross (ROSS, 2012), i.e., through the generation of independent exponential random variables, each with rate λ.

On the other hand, the penalty time represents a truncated Erlang random variable that may assume, in some cases, the value of zero when $Y_i < \tau$. Algorithm 1 describes the steps to calculate the penalty time incurred by the agent.

Algorithm 1 – Simulation of the penalty time
• **Input**: λ, μ, L, τ • **Output**: The number of product failures and the penalty time.

Step 1: Set $T = 0$ //This variable sums the exponential random numbers (failure times).
Step 2: Set $N = 0$ //This variable counts the number of exponential random numbers generated (number of product failures).
Step 3: Loop over the exponential random numbers.
a. $while\ (1)\{$ //Infinite loop.
i. $X \sim Exp(\lambda)$; //Time between failures.
ii. $T = T + X$;
iii. if$(T > L)\{break\ out\}$; //The loop is immediately terminated – break statement.
iv. $N = N + 1$;}
Step 4: To generate Y N times //Y is the agent's repair time, being $Y \sim Exp(\mu)$.
Step 5: To compare each Y with τ:
a. *if* $(Y \geq \tau)$; *then store*
b. *otherwise assign 0*.
Step 6: To create a vector with these differences and store them.
Step 7: To sum this vector.

Source: This research

Algorithm 1 covers one replication. It produces one penalty time conditioned by the simulation result of $N(L)$. Calculate $\mathrm{E}_{N(L)}\left[\mathrm{E}_{Y_i - \tau}\left[\sum_{i=0}^{N(L)} max(0, Y_i - \tau)\,|N(L)\right]\right]$ is necessary to perform the loop of Algorithm 1 and compute the average value with all realizations.

2.5.3.2 Consumer's warranty cost after W_2 - Option O_3

This simulation calculates the consumer's warranty cost after W_2. The procedure simulates a homogeneous Poisson process that incorporates failure times after W_2, with the equipment's sale price (P_E) and the lifetime (L).

Algorithm 2 describes its steps. In broad words, it computes the consumer's warranty cost for each failure until L. Finally, it sums all these costs, producing the total cost incurred by the consumer after W_2.

Algorithm 2 – Simulation of the consumer's warranty cost after W_2
• **Input**: λ, L, W_2, P_E • **Output**: The consumer's warranty cost incurred by the consumer after W_2.

Step 1: Set $T = W_2$ //The variable sums exponential random numbers generated after a period of length W_2 (failure times).
Step 2: Set $SMC = 0$ //This variable provides the consumer's warranty cost after W_2.
Step 3: Loop over the exponential random numbers.
a. $while(1)\{$ //Infinite loop.
i. $X \sim Exp(\lambda)$; //Time between failures.
ii. $T = T + X$;
iii. if$(T > L)\{break\ out\}$; //The loop is immediately terminated – break statement.
iv. $C = P_E\left(\frac{L-T}{L}\right)$; //The consumer's cost for each failure.
v. $SMC = SMC + C;\}$
Step 4: $Print\ SMC$.

Source: This research

Such as in Algorithm 1, Algorithm 2 provides as output a realization, the consumer's warranty cost after W_2. Calculate $P_E \mathrm{E}_{N(L-W_2)}\left[\mathrm{E}_{T_i}\left[\left(\sum_{i=0}^{N(L-W_2)} \frac{T_i}{L}\right) - N(L-W_2) | N(L-W_2)\right]\right]$ is necessary to perform the loop of Algorithm 2, calculating the sample mean with all realizations.

2.6 Application example

2.6.1 Model's parameters

A numerical application is carried out to illustrate what is the game equilibrium. The following nominal values for the model's parameters are: $L = 40{,}000$

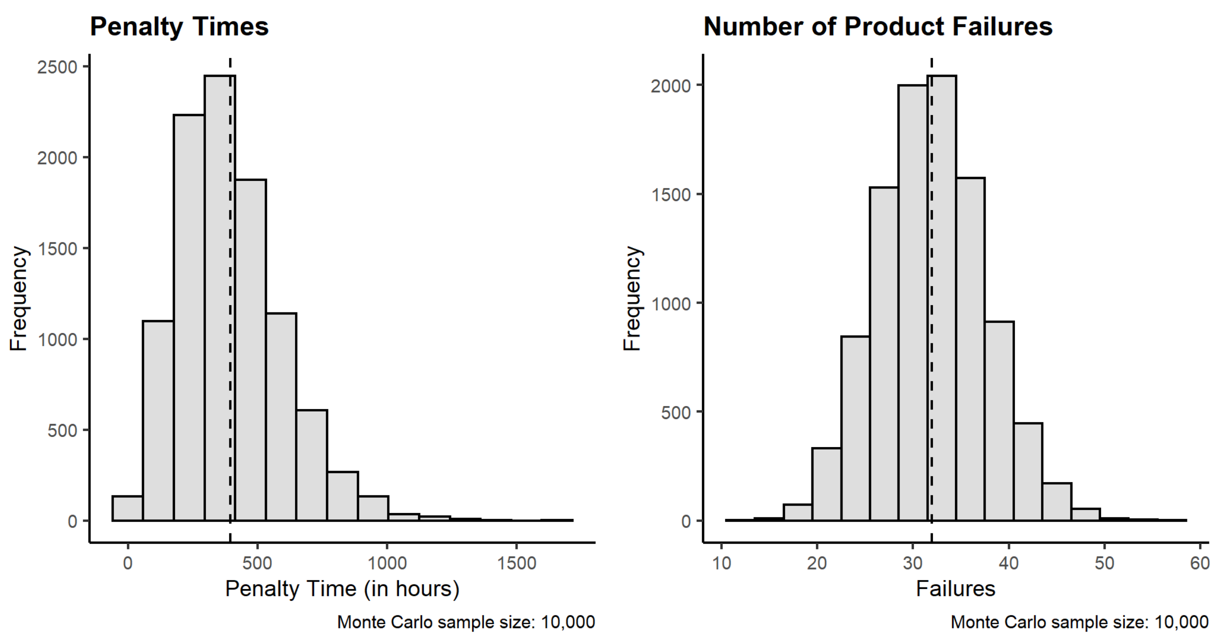
(hours), $\lambda = 0.0008$ (per hour), $\mu = 0.02$ (per hour), $\tau = 70$ (hours), $P_E = 300(10^3\$)$, $\alpha = 0.06(10^3\$)$, $R = 0.015(10^3\$$ per hour), $\xi = 6(10^3\$)$, $C_A = 5(10^3\$)$, $W_1 = 20{,}000$ (hours), and $W_2 = 30{,}000$ (hours).

2.6.2 Simulation results

We used Ox, a matrix programming language with object-oriented support developed by Jurgen Doornik (CRIBARI-NETO; ZARKOS, 2003), to carry out the Monte Carlo simulations; the number of replications was 10,000.

Algorithm 1 provides as output the penalty time for each $N(L)$. Two simulations are carried out sequentially. The resume of the simulation results can be seen in Figure 5 and Table 3.

Figure 5 – Histograms of simulation results – Option O_1



Source: This research

The dashed lines presented in Figure 5 are the expected values of the 10,000 Monte Carlo replicas. For the penalty time, its expected value is 395.11 hours, whereas the number of product failures is 31.93, close to the theoretical mean of the homogeneous Poisson process ($\mathrm{E}[N(L)] = 32$).

We mention that the shape of the histogram related to the number of product failures (Figure 5) is close to the normal distribution. This result comes from the central limit theorem. It states when the random samples are sufficiently large, the distribution of the sample means will be asymptotically normal (ROSS, 2014).

Table 3 – Descriptive statistics of simulation results – Option O_1

Variables	**Mean**	**Median**	**Max**	**Min**	**Standard Deviation**	**Skewness**	**Kurtosis (excess)**
Penalty Time	395.11	368.84	1656.23	0	200.17	0.83	1.08
Number of Failures	31.93	32	58	13	5.65	0.18	0

Source: The author

Concerning the results of the simulations of the penalty time, 5 replicas presented that the penalty time is equal to 0, whereas 78 replications exhibited a penalty time above 1000 hours. Furthermore, Table 4 shows a nonparametric confidence interval for the expected value of the penalty time via the bootstrap approach (EFRON, 1979).

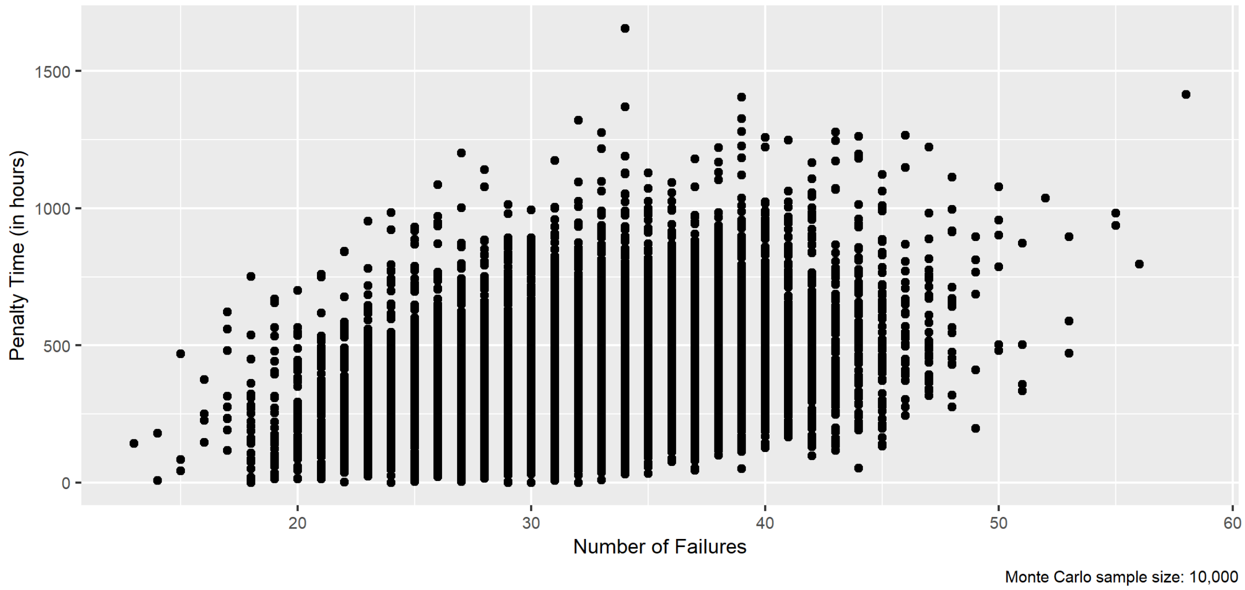
Table 4 – Confidence interval for the expected value of the penalty time – 95% confidence level

Variable	**Mean**	**Lower Endpoint**	**Upper Endpoint**
Penalty Time	395.11	391.65	398.39

Source: The author

Figure 6 agglutinates the simulation results of the penalty time and the number of product failures in a scatter plot. Spearman's rank correlation coefficient (CORDER; FOREMAN, 2014) is 0.35 ($p < 0.05$), implying that the strength and direction of association between the penalty time and the number of product failures are moderate. As a result, there is a positive and medium monotonic relationship between such variables.

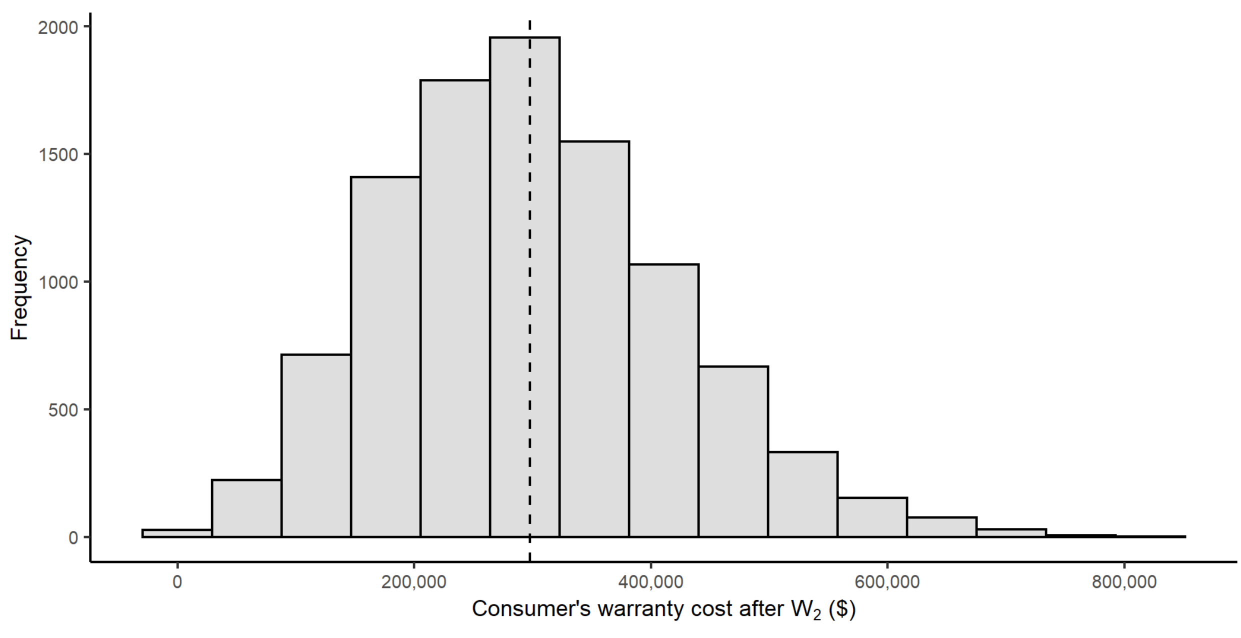
Figure 6 – Scatter plot – relationship between the penalty time and the number of product failures



Source: This research

The summary of the simulation results related to Algorithm 2 (consumer's warranty cost after W_2) can be seen in Figure 7 and Table 5.

Figure 7 – Histogram of simulation results – Option O_3



Source: This research

The dashed line presented in Figure 7 is the consumer's expected warranty cost after W_2. It corresponds to $297.901\ (10^3\$)$.

Table 5 – Descriptive statistics of simulation results – Option O_3

Variables	Mean	Median	Max	Min	Standard Deviation	Skewness	Kurtosis (excess)
Consumer's warranty cost	297,901	289,867	821,836	0	121,698	0.45	0.16

Source: The author

In 5 simulations, the consumer's warranty cost was 0 after the W_2 period. In contrast, in 150 simulations, the consumer's warranty cost was above the revenue due the use of the product, $600(10^3\$)$.

Finally, table 6 shows a nonparametric confidence interval for the expected value of consumer's warranty cost after W_2 via the bootstrap approach.

Table 6 – Confidence interval for the expected value of the consumer's warranty cost after W_2– 95% confidence level

Variable	Mean	Lower Endpoint	Upper Endpoint
Consumer's warranty cost	$297.901(10^3\$)$	$295.861(10^3\$)$	$299.839(10^3\$)$

Source: The author

2.6.3 Analysis of results

The consumer's reservation prices can be seen in Table 7. If the agent sets up all their prices equal to them, the consumer is indifferent among the warranty options since their expected profit is zero. Under this context, the agent obtains as expected profit $440(10^3\$)$, extracting all consumer surplus.

Concerning warranty option O_3 (Hybrid maintenance contract), we emphasize that $\overline{PW}_2$ is very close to the consumer's expected warranty cost W_2. On the other hand, the coverage period W_2 is three times more than the equipment's residual lifetime $(L - W_2)$.

Table 7 – Consumer's reservation prices

Consumer's reservation prices	$\overline{P}_M$	$\overline{P}_R$	$\overline{PW}_2$	$\overline{P}_T$
	$623.707(10^3\$)$	$31.5(10^3\$)$	$302.099(10^3\$)$	$18.75(10^3\$)$

Source: The author

2.6.4 Sensitivity analysis

A sensitivity analysis is performed to investigate the consumer's reservation prices, the average penalty time, the expected value of the number of product failures, the consumer's expected warranty cost after W_2, and the agent's expected profit for different values of λ. The summary of these results can be seen in Table 8 and Figure 8.

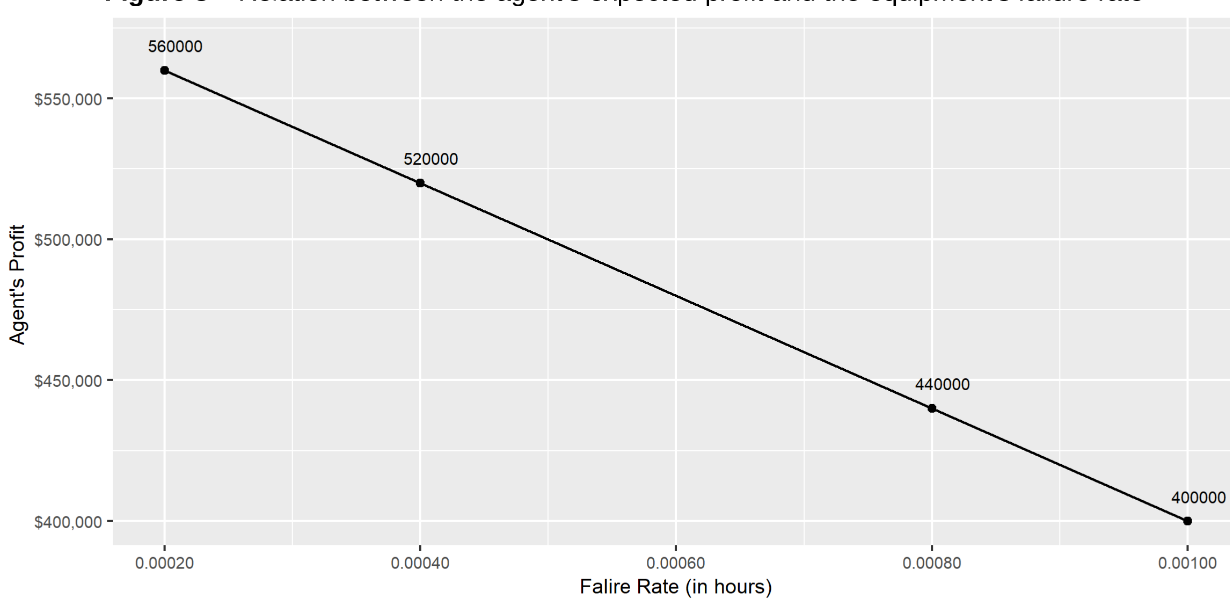
Table 8 – Effect of λ on reliability-related performance measures and the consumer's reservation prices

	$\boldsymbol{\lambda}$			
	$\boldsymbol{\lambda = 0.0002}$	$\boldsymbol{\lambda = 0.0004}$	$\boldsymbol{\lambda = 0.0008}$	$\boldsymbol{\lambda = 0.001}$
$\mathbf{E}[\boldsymbol{N(L)}]$	8	16	32	40
$\mathbf{E}[\boldsymbol{Penalty\ time}]$	98.93 hours	197.99 hours	395.11 hours	492.19 hours
$\mathbf{E}\left[\begin{matrix}\boldsymbol{Consumer's\ warranty} \\ \boldsymbol{after\ W_2}\end{matrix}\right]$	$74.909(10^3\$)$	$149.637(10^3\$)$	$297.901(10^3\$)$	$373.905(10^3\$)$
$\boldsymbol{\bar{P}_M}$	$605.936(10^3\$)$	$611.879(10^3\$)$	$623.707(10^3\$)$	$629.531(10^3\$)$
$\boldsymbol{\bar{P}_R}$	$144(10^3\$)$	$69(10^3\$)$	$31.5(10^3\$)$	$24(10^3\$)$
$\boldsymbol{\overline{PW}_2}$	$525.091(10^3\$)$	$450.364(10^3\$)$	$302.099(10^3\$)$	$226.095(10^3\$)$
$\boldsymbol{\bar{P}_T}$	$75(10^3\$)$	$37.5(10^3\$)$	$18.75(10^3\$)$	$15(10^3\$)$

Source: The author

Table 8 shows that $\mathrm{E}[Penalty\ Time]$, $\mathrm{E}[Consumer's\ warranty\ cost\ after\ W_2]$, $\mathrm{E}[N(L)]$, and $\bar{P}_M$ increase and $\bar{P}_R$, $\overline{PW}_2$, and $\bar{P}_T$ decrease as λ increases. Furthermore, when the equipment's reliability decreases (λ increases), the agent's expected profit also reduces (Figure 8). Thus, when the reliability is low, the agent's gain decreases since their costs increase.

Figure 8 – Relation between the agent's expected profit and the equipment's failure rate



Source: The author

2.6.5 Extension

In this extension, we consider the situation when the consumer does know the correct value of λ in advance, being discovered along with L. Thus, the equipment owner builds two scenarios to the failure rate, optimistic and pessimistic. The optimistic scenario represents the situation that the failure rate is below the correct value, while the pessimistic scenario is the opposite case.

Let λ_1 denote the precise value of λ. The consumer estimates $\lambda_2(<\lambda_1)$ (the optimistic scenario of λ) with probability q and $\lambda_3(>\lambda_1)$ (the pessimistic scenario of λ) with probability $1-q$. Furthermore, the agent knows q. The nominal values are $\lambda_1 = 0.0008$ (per hour), $\lambda_2 = 0.0004$ (per hour), $\lambda_3 = 0.001$ (per hour), and $q = 0.7$. The other model parameters are the same that the numerical example.

If the agent defines their equilibrium strategy through the expected value solution (BIRGE; LOUVEAUX, 2011), their pricing represents a weighted mean of the consumer reservation prices whose weights are q and $1-q$. We found that $\dot{p}_A = \left(P_M = 617.175\ (10^3\$), P_R = 55.550\ (10^3\$), PW_2 = 383.068\ (10^3\$), P_T = 30.750\ (10^3\$)\right)$ and $\dot{x}(\dot{p}_A) = 1$. The agent's expected payoff is $433.468(10^3\$)$, whereas the consumer's expected payoff is $6.532(10^3\$)$.

Now, we consider a slight variation, when both scenarios (optimistic and pessimistic) present the failure rate above the correct value of $\lambda(=\lambda_1)$, and $\lambda_2 < \lambda_3$. Thus, through these nominal values $\lambda_1 = 0.0002$ (per hour), $\lambda_2 = 0.0004$ (per hour), $\lambda_3 = 0.0008$ (per hour), and $q = 0.7$, the equilibrium is $\dot{p}_A = \left(P_M = 615.427(10^3\$), P_R = 57.750\ (10^3\$), PW_2 = 405.884\ (10^3\$), P_T = 31.875\ (10^3\$)\right)$ and $\dot{x}(\dot{p}_A) = 2 \text{ or } 4$. For this context, only option O_1 provides a negative profit for the consumer. The agent's expected profit is $215(10^3\$)$ whereas the consumer's expected payoff is $345(10^3\$)$.

Therefore, under certain circumstances, the consumer may obtain a positive payoff. Such a fact occurs when the agent does not define the consumer's reservation prices correctly. Note that different perspectives around the equipment's failure rate led to distinct equilibrium strategy profiles.

2.7 Concluding and remarks

This research presented a warranty decision problem between a consumer, an owner of a repairable system, and an agent who carries out the maintenance services. The agent shows four warranty options, with different coverage designs for the equipment's lifetime.

The model is a sequential game, and the equilibrium strategies are given via the sub-game perfect Nash equilibrium for each player. The equilibrium prices were developed using the consumer's reservation prices that permit the agent to extract all consumer surplus.

Another characteristic described in this modeling is the adoption of the computer simulation that provides estimates of the warranty costs for some warranty options for whose mathematical functions cannot be evaluated smoothly. Under this perspective, we analyze the long-term behavior of some reliability-related performance measures such as the penalty time and the consumer's warranty cost after W_2.

The sensitivity analysis focused on the consumer's reservation prices adjusts as the failure rate changes. As a result, the agent's expected profit also switches. Furthermore, the extension presented is associated with when the consumer does not know the correct value of λ in advance, providing a chance to the product owner to obtain a positive payoff once the agent does not define the consumer's reservation prices precisely. Under this context, the agent sets up their equilibrium prices via the expected value solution.

The model can be further extended in several ways to evaluate other metrics, whether in the game theory formulation or a reliability perspective. We suggest eight possibilities, as follows:

1. To introduce a utility function with a risk parameter to model the consumer's decision regarding the warranty option to be chosen, as seen in (ASGHARIZADEH; MURTHY, 2000; ESMAEILI; SHAMSI GAMCHI; ASGHARIZADEH, 2014; MURTHY; ASGHARIZADEH, 1998).
2. To extend this decision problem by including other parties. For example, more than one consumer (ASGHARIZADEH; MURTHY,

2000) or another party to carry out the maintenance services, considering a competition model (KAMESHWARAN et al., 2009; KURATA; NAM, 2010).

3. This work holds an assumption related to the homogeneous Poisson process as a stochastic process that models the failure-repair-failure cycle of the product (assumption 3). We could apply another stochastic process such as a non-homogeneous Poisson, as seen in (ESMAEILI; SHAMSI GAMCHI; ASGHARIZADEH, 2014; JACKSON; PASCUAL, 2008). Hence, we can compare the equilibrium strategies and reliability-related performance measures for both cases.
4. Assumption 3 also states that the agent's repair times are infinitesimal concerning mean times between failures. Thus, the summation of repair times is negligible, and the consumer's financial revenue is approximately RL. Without this assumption, the failure-repair-failure cycle of the product cannot be modeled as a point process since the repair times will affect the consumer's revenue. Therefore, this novel framework leads to a new concept of simulation that can be analyzed deeper in future research.
5. The agent’s strategy is to define pricing for the warranty options. We could add other variables to their strategy set. For instance, W_1 and W_2 can be seen as decision variables for the agent.
6. The model assumes that when the agent practices the consumer's reservation prices to set up their equilibrium strategy, the consumer is indifferent among the warranty options since their expected payoff is equal to zero. An interesting situation would be when the consumer's equilibrium strategy is not unique. Whether two or more options provide the same expected gain to the consumer, but the agent's payoff changes with them, the model becomes more complex.
7. This work can be modeled as a warranty economic decision problem considering the agent's costs (or warranty reserves) over a time interval. We set up four warranty options with different features of

coverage. As a result, research can be addressed whose motivation is to construct cost models for warranties, analyzing economic viability. Menke (MENKE, 1969) developed a similar study considering a non-repairable system.

8. We could consider a discounted cost for the players' payoff functions, similar to Lam and Kwok Wai Lam's (LAM; KWOK WAI LAM, 2001) and Mamer's (MAMER, 1987) studies. A suggestion is to apply the discount for the consumer's payoff function related to warranty option O_4 (services on demand) since the consumer deals with unrecoverable costs such as the transaction costs in invoking the warranty service from the agent.
9. The second model of this thesis analyzes this problem under a different view, i.e., pricing of warranty services when two parties carry out maintenance services along with the equipment's lifetime. As a result, how to define an equilibrium price strategy when two decision-makers must define their pricing. We bring cooperative games to solve this decision problem.

"*I, wisdom, dwell together with prudence;*
I possess knowledge and discretion. To fear the Lord is to hate evil;
I hate pride and arrogance, evil behavior and perverse speech."
Proverbs 8:12-13

3 SECOND ESSAY. HIERARCHICAL GAME: AN APPLICATION OF A THREE-PERSON GAME IN THE PRICING OF THE PRODUCT WARRANTY

This paper is a hierarchical game defined to a manufacturer, an agent, and a consumer applied to the product warranty. Through a mix of non-cooperative and cooperative solutions, equilibrium strategies are developed. The model brings a coalition between the manufacturer and the agent. The manufacturer defines the sale price of the product (including the base warranty costs), whereas the agent prices maintenance services. Finally, the Shapley value redistributes the total gain between the participants, assigning the equilibrium prices.

Keywords: Hierarchical game. Warranty. Maintenance. Price strategy. Shapley value.

3.1 Highlights

- A new concept of equilibrium is presented, characterized by the coordination of strategies between the manufacturer and the agent.
- We detail all characteristic functions between the coalition established - manufacturer and agent.
- We use computer simulation to estimate the expected value of maintenance services incurred by the agent.
- A sensitivity analysis to evaluate the influence of the model parameters on the equilibrium strategies and players' payoff.

3.2 Model's motivation

According to Esmaeili et al. (ESMAEILI; SHAMSI GAMCHI; ASGHARIZADEH, 2014), there is a significant shortcoming for warranty problems that include the manufacturer, the agent, and the consumer (product owner). This omission is uncommon since those most products these days, particularly high-tech products, are supported by a third-party warranty in addition to the manufacturer's warranty.

Empirically, SquareTrade[3] sells protection plans to many appliances and tech products (iPad, MacBook, iPhone, Smart Watch) considering several manufacturers (Apple, Samsung Galaxy). As a result, the product owner can obtain both coverages for their item for different periods.

Therefore, this model aims to fill this gap under a theoretical view. In equilibrium, we show a coalition between the manufacturer and the agent that coordinates their pricing strategy to extract the consumer surplus.

3.3 Originality

Only three papers devoted their attention to model a three-level service contract based on an integrated approach (ESMAEILI; SHAMSI GAMCHI; ASGHARIZADEH, 2014; GAMCHI; ESMAEILI; MONFARED, 2013; TALEIZADEH; SHERAFATI, 2019). Our difference from them is associated with the game setting and the composition of the equilibrium strategies. We focus on the pricing of the warranty.

The manufacturer prices equipment (including the base warranty costs), and the agent determines maintenance services. Equilibrium strategies are via a mix of non-cooperative and cooperative solutions.

The cooperative approach shows a coalition formed between the manufacturer and the agent. The Shapley value redistributes the collective payoff and defines the equilibrium prices. The use of the Shapley value is that it provides a unique solution (SUN; SUN, 2018), a singular price pair.

Otherwise, the non-cooperative solution is based on the subgame-perfect Nash equilibrium due to the sequential structure of the model. The manufacturer and the agent offer their prices while the consumer replies. We assume that the product owner does not abandon the product after the base warranty period. As a result, the consumer decides whether to buy or not buy the product with maintenance services.

The decision problems in previous papers (ESMAEILI; SHAMSI GAMCHI; ASGHARIZADEH, 2014; GAMCHI; ESMAEILI; MONFARED, 2013; TALEIZADEH; SHERAFATI, 2019) were formulated considering different possibilities of equilibrium for

[3] More details about SquareTrade can be seen at https://www.squaretrade.com/.

multiple scenarios and other decision variables besides prices, such as the warranty period and or marketing expenditure.

3.4 Notation list

3.4.1 Input parameters

L: equipment's lifetime;

L_1 base warranty period;

λ: failure rate;

μ: repair rate;

Y: agent's repair time;

τ: threshold time to finish the maintenance without penalty;

C_P: manufacturing cost;

C_{RM}: manufacturer's maintenance cost;

C_A: agent's repair cost;

α: penalty cost per time;

$\mathrm{E}[\Pi_M(.)]$: manufacturer's payoff function;

$\mathrm{E}[\Pi_A(.)]$: agent's payoff function;

$\mathrm{E}[\Pi_C(.)]$: consumer's payoff function;

$N(L_1)$: number of product failures over L_1;

$N(L - L_1)$: number of product failures between L_1 and L;

S_{HM}: manufacturer's expected payoff in equilibrium;

S_{HA}: agent's expected payoff in equilibrium;

$V(M)$: manufacturer's characteristic function;

$V(A)$: agent's characteristic function;

$V(MA)$: grand coalition's characteristic function;

R: revenue.

3.4.2 Decision variables

P_E: equipment's sale price;

P_{MC}: price of the maintenance service contract.

3.4.3 Equilibrium prices

$\bar{P}_E$: equipment's sale price of equilibrium;

$\bar{P}_{MC}$: equilibrium price of the maintenance service contract.

3.5 Model Formulation

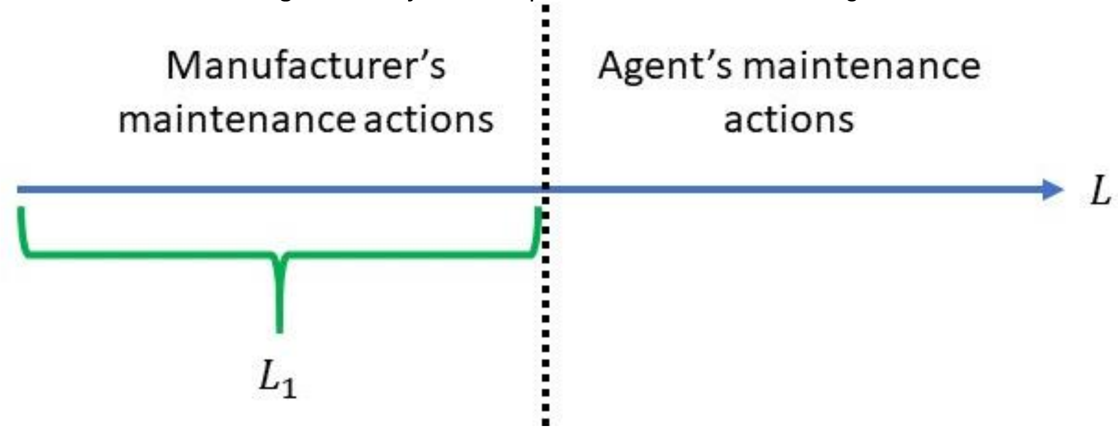
The model is a three-person game, defined between the manufacturer, the agent, and the consumer. The manufacturer sells a durable good (repairable system), including the base warranty whose price is P_E to the consumer. After the base warranty period, the agent provides product maintenance to the residual lifetime of the good.

The product generates revenue R per unit time in the working state during its lifetime (L) to the product owner. In contrast, in the failed state, the revenue is zero. The item receives maintenance from the manufacturer or the agent, depending on the lifetime period.

The design of product maintenance differs between the manufacturer and the agent, although both were classified as free replacement warranty policies (THOMAS; RAO, 1999). In $L_1(< L)$, the base warranty period, the manufacturer carries out maintenance. After the failure, the manufacturer repairs the item.

Posteriorly, the agent carries out the maintenance for the residual lifetime of the product. For a fixed P_{MC}, the agent agrees to repair all failures over $[L_1, L)$ at no additional cost. If the repair is not finished until time τ, the agent incurs a penalty time. Let Y denote the agent's repair time, and α denote the penalty cost per time, expressed in monetary units. The penalty design is $\alpha(Y - \tau)$, if $Y > \tau$, and zero otherwise. Under this context, the penalty represents financial compensation that may be received by the consumer related to the downtime to recover the device when the repair time is extensive. Figure 9 summarizes the dynamic of product maintenance during L, considering who carries out over L.

Figure 9 – Dynamic of product maintenance during L.



Source: The author

3.5.1 Equipment failures and repairs

Estimate the possible costs regarding product maintenance incurred by the manufacturer and the agent; the failure frequency should be estimated first. Concerning the failure rate $(\lambda > 0)$, the failure-repair-failure cycle of the product is assumed to be a homogeneous Poisson process. In this stochastic process, the times between failures are independent and identically exponential random variables with the same parameter (GNEDENKO; USHAKOV, 1995). This characterization is appropriate when the product is in the second part of the bathtub curve, where failures are assumed to occur randomly, and the failure rate is a constant (BALACHANDRAN; MASCHMEYER; LIVINGSTONE, 1981).

Under a homogeneous Poisson process, the expected value of the number of product failures in the period $[0, w]$, where $w(> 0)$ is a prefixed time, is λw, following a Poisson distribution.

We also assume that the agent's repair time also follows an exponential distribution with repair rate $\mu(> 0)$.

Since the failure rate is constant, there is no need for preventive maintenance (MURTHY; ASGHARIZADEH, 1998). Thus, the manufacturer and the agent only provide corrective maintenance. After the failure, repair actions are carried out.

3.5.2 Manufacturer's decision problem

As the coverage period of the base warranty is fixed, being L_1. The manufacturer prices the product in P_E. Note that the base warranty costs are factored into the product price. Therefore, the manufacturer's expected profit is given by

$$\mathrm{E}[\Pi_M(P_E)] = P_E - C_P - C_{RM}\lambda L_1, \tag{3.1}$$

where C_P is the manufacturing costs to make the product, and C_{RM} is the manufacturer's maintenance cost for each intervention in the base warranty period.

3.5.3 Agent's decision problem

The agent is only responsible for product maintenance between L_1 and L. Hence, the agent prices the maintenance service contract in P_{MC}. Therefore, the agent's expected profit is given by

$$\mathrm{E}[\Pi_A(P_{MC})] = P_{MC} - \alpha \mathrm{E}_{N(L-L_1)}\left[\mathrm{E}_{Y_i-\tau}\left[\sum_{i=0}^{N(L-L_1)} max(0, Y_i - \tau)\,|N(L - L_1)\right]\right] - C_A\lambda(L - L_1), \tag{3.2}$$

where $\mathrm{N}(L - L_1)$ is a random variable that indicates the number of product failures over $[L_1, L)$, and C_A is the agent's repair cost for each intervention.

The expression $\alpha \mathrm{E}_{N(L-L_1)}\left[\mathrm{E}_{Y_i-\tau}\left[\sum_{i=0}^{N(L-L_1)} max(0, Y_i - \tau)\,|N(L - L_1)\right]\right]$ represents the expected value of the financial compensation to be received by the consumer related to the penalty time. Formally, this double expectation is the expected value of the penalty time by conditioning it on the number of product failures after L_1. It formulates under the concept of the law of total expectation from conditional expectation (ROSS, 2010), i.e., the number of product failures affects the penalty time.

3.5.4 Consumer's decision problem

The consumer is interested in purchasing a single unit of the good in question. We assume that after the base warranty period, the consumer does not abandon the item after the failure, i.e., the product owner buys the maintenance services from the agent. Therefore, the consumer's expected profit is given by

$$\mathrm{E}[\Pi_C(P_E, P_{MC})] = RL + \alpha \mathrm{E}_{N(L-L_1)}\left[\mathrm{E}_{Y_i-\tau}\left[\sum_{i=0}^{N(L-L_1)} max(0, Y_i - \tau) \,|N(L - L_1)\right]\right] - P_E - P_{MC}. \quad (3.3)$$

3.5.5 Assumptions

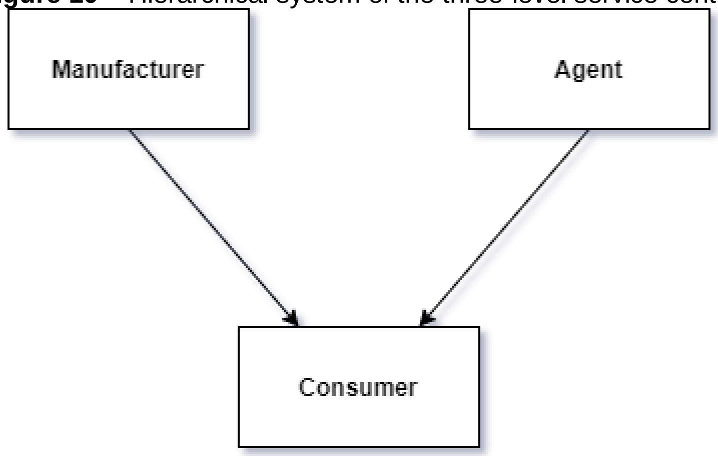
1. All decision-makers seek to maximize their payoff functions. Furthermore, the consumer's satisfaction is maximized by buying the product with maintenance services.
2. All elements of the game are known (complete information).
3. The repair times (manufacturer and agent) are infinitesimal concerning mean times between failures, being negligible compared to the product's lifetime. Moreover, under the homogeneous Poisson process, the type of repair is perfect. Hence, after each repair, the condition of the repaired item is assumed to be as good as new (BLISCHKE; MURTHY, 1994).
4. The repair costs (labor + material) incurred by the manufacturer and the agent do not change over L.

According to Assumption 3, repair times are negligible compared to the equipment's lifetime. However, these times affect the penalty time incurred by the agent. This assumption also is seen in (ASGHARIZADEH; MURTHY, 2000; MURTHY; ASGHARIZADEH, 1998, 1999).

3.6 Game setting and equilibrium

The model is a hierarchical system with double subordination, similar to the diamond-shape (PETROSYAN; ZENKEVICH, 2016). Formally, the consumer's strategy is subordinated to the manufacturer's and the agent's strategies. The payoff for all decision-makers depends only on the consumer's decision that is a function of P_E and P_{MC}. Figure 10 gives a visual description of the model. The arrows from the manufacturer and the agent toward to consumer imply the dependence relationship.

Figure 10 – Hierarchical system of the three-level service contract



Source: The author

The model is a sequential two-stage game with complete information. In the upper level, the manufacturer selects a value to P_E, and the agent chooses a value to P_{MC}. Next, based on these prices, the consumer takes a decision.

The consumer does not obtain the item if their expected payoff is negative, indifferent between if their expected profit is zero. Finally, if their expected gain is positive, the consumer acquires the product with maintenance services.

The equilibrium is reached via a mix of cooperative and non-cooperative solutions. For every fixed pair (P_E, P_{MC}), the consumer gives a reply, evaluating their payoff function to decide whether to buy the product with maintenance services or not. For this part, equilibrium strategies can be formulated via the subgame-perfect Nash equilibrium (MAZALOV, 2014).

At first, the manufacturer and the agent deal with a pricing problem. As these decision-makers are at the same level, we can establish a coalition between them. The equilibrium prices and the payoffs are computed via the Shapley value. The Shapley value is an allocation scheme that provides a unique solution to the coalition, measuring the marginal contribution for each decision-maker (SUN; SUN, 2018), holding efficiency, symmetry, null player, and additivity properties (MASCHLER; SOLAN; ZAMIR, 2013).

3.6.1 Characteristic functions

Before calculating the Shapley, it is necessary first to set up the characteristic function for each sub-coalition. For two players, there are four sub-coalitions: grand coalition (manufacturer and agent), empty coalition (no player), manufacturer, and agent.

3.6.1.1 Manufacturer's characteristic function

The payoff received by the manufacturer is P_E. We define their characteristic function as the least price that the manufacturer is willing to sell the product, including warranty costs during period L_1, i.e., break-even price (SAMUELSON; MARKS, 2014). Therefore, the manufacturer's characteristic function is given by

$$\mathrm{V}(M) = C_P + C_{RM}\lambda L_1, \tag{3.4}$$

where $\mathrm{V}(M)$ is the manufacturer's characteristic function.

3.6.1.2 Agent's characteristic function

The payoff received by the agent is P_{MC}. We define their characteristic function as the least price that the agent is willing to offer to the consumer for maintenance services, i.e., the penalty and repair costs. Therefore, the agent's characteristic function is given by

$$\mathrm{V}(A) = \alpha \mathrm{E}_{N(L-L_1)}\left[\mathrm{E}_{Y_i-\tau}\left[\sum_{i=0}^{N(L-L_1)} max(0, Y_i - \tau)\,|N(L-L_1)\right]\right] + C_A\lambda(L - L_1), \tag{3.5}$$

where $\mathrm{V}(A)$ is the agent's characteristic function.

3.6.1.3 Grand coalition's characteristic function

The characteristic function of the grand coalition is defined when the consumer surplus is zero due to the summation of the manufacturer's and agent's prices (P_E, P_{MC}). As a result, its payoff is the maximum total profit available.

The grand coalition's characteristic function is given by

$$\mathrm{V}(M, A) = RL - C_P - C_{RM}\lambda L_1 - C_A\lambda(L - L_1), \tag{3.6}$$

where $\mathrm{V}(M, A)$ is the characteristic function of the grand coalition formed by the manufacturer and the agent.

3.6.1.4 Empty coalition's characteristic function

Since that the empty coalition does not include any decision-maker, its payoff is zero.

3.6.2 Shapley value

Through the Shapley value, the marginal contribution for each decision-maker is calculated, representing the equilibrium payoff. Let S_{HM} be the manufacturer's expected payoff and S_{HA} be the agent's expected payoff, where

$$S_{HM} = \frac{V(M)+V(M,A)-V(A)}{2}, \quad (3.7)$$

$$S_{HA} = \frac{V(A)+V(M,A)-V(M)}{2}. \quad (3.8)$$

3.6.3 Equilibrium prices

The manufacturer's and the agent's equilibrium prices are $\bar{P}_E$ and $\bar{P}_{MC}$, respectively. $\bar{P}_E$ is defined via $Eq. 3.1$ and $Eq. 3.7$, and $\bar{P}_{MC}$ is given by $Eq. 3.2$ and $Eq. 3.8$.

$$\bar{P}_E = S_{HM} + C_P + C_{RM}\lambda L_1, \quad (3.9)$$

$$\bar{P}_{MC} = S_{HA} + \alpha \mathrm{E}_{N(L-L_1)}\left[\mathrm{E}_{Y_i-\tau}\left[\sum_{i=0}^{N(L-L_1)} max(0, Y_i - \tau)\,|N(L-L_1)\right]\right] + C_A\lambda(L - L_1). \quad (3.10)$$

Under this context, the summation between $\bar{P}_E$ and $\bar{P}_{MC}$ is equal to the consumer surplus. This cooperation (manufacturer and agent) is like a monopolist applying first-degree price discrimination (VARIAN, 1989).

3.7 Simulation approach

The simulation is applied to find an estimator to the expected value of the penalty time incurred by the agent. Its procedure is similar to Algorithm 1 (Chapter 02). This simulation deals with two random variables, the number of product failures after L_1 and the penalty time.

In broad words, the simulation of the number of product failures follows the approach seen in Ross (ROSS, 2012). Thus, failure times are generated after L_1, and the number of claims is saved. For each product failure, a repair time is produced and compared with τ. If the time difference is positive, then store it, otherwise assign zero. The summation of the penalty times represents the total penalty time. Algorithm 3 shows the steps of the simulation.

Algorithm 3 – Simulation of the penalty time

- **Input**: $\lambda, \mu, L, \tau, L_1$
- **Output**: The number of product failures and the penalty time.

Step 1: Set $T = L_1$ //This variable sums the exponential random numbers (failure times).
Step 2: Set $N = 0$ //This variable counts the number of exponential random numbers generated (number of product failures).
Step 3: Loop over the exponential random numbers.
  a. *while* $(1)\{$ //Infinite loop.
    i. $X \sim Exp(\lambda)$; //Time between failures.
    ii. $T = T + X$;
    iii. if$(T > L)\{break\ out\}$; //The loop is immediately terminated – break statement.
    iv. $N = N + 1;\}$
Step 4: To generate Y N times //Y is the agent's repair time, being $Y \sim Exp(\mu)$.
Step 5: To compare each Y with τ:
  a. *if* $(Y \geq \tau)$; *then store*
  b. *otherwise assign 0.*
Step 6: To create a vector with these differences and store them.
Step 7: To sum this vector.

Source: This research

Note that Algorithm 3 computes one replica. Therefore, to estimate the expected value of the penalty time, we can use the Monte Carlo approach (TAHA, 2016), performing the loop of Algorithm 3.

3.8 Application example

3.8.1 Model's parameters

A numerical application is carried out to illustrate what is the game equilibrium. The following nominal values for the model's parameters are: $L = 45{,}000$ (hours), $L_1 = 8{,}500$ (hours), $\lambda = 0.0008$ (per hour), $\mu = 0.02$ (per hour), $\tau = 70$ (hours), $C_P = 300(10^3\$)$, $\alpha = 0.06(10^3\$)$, $R = 0.015(10^3\$$ per hour), $C_A = 5(10^3\$)$, and $C_{RM} = 4(10^3\$)$.

3.8.2 Analysis of results

Table 9 presents the expected value of the reliability-related performance measures. The Monte Carlo sample size for simulating the penalty time was 10,000.

Table 9 – Expected value of the reliability-related performance measures

$\boldsymbol{E[N(L_1)]}$	$\boldsymbol{E[N(L-L_1)]}$	$\boldsymbol{E[Penalty\ Time]}$
6.8	29.2	357.82 (hours)

Source: The author

Furthermore, $\bar{P}_E = 507.965(10^3\$)$ and $\bar{P}_{MC} = 188.504(10^3\$)$. Through these equilibrium prices, the manufacturer's expected profit is $180.765(10^3\$)$, and the agent's expected profit is $21.035(10^3\$)$. Based on this price combination, the manufacturer and the agent extract the consumer surplus. In equilibrium, the consumer is indifferent between purchasing and not the product with maintenance services.

3.8.3 Sensitivity analysis

A sensitivity analysis is performed to investigate the equilibrium prices, the average penalty time, the expected value of the number of product failures, and the manufacturer's and the agent's expected profit for different values of λ. The summary of these results can be seen in Table 10.

Table 10 – Effect of λ on the model

	$\boldsymbol{\lambda}$			
	$\boldsymbol{\lambda = 0.0002}$	$\boldsymbol{\lambda = 0.0004}$	$\boldsymbol{\lambda = 0.0008}$	$\boldsymbol{\lambda = 0.001}$
$\mathbf{E}[N(L_1)]$	1.7	3.4	6.8	8.5
$\mathbf{E}[N(L-L_1)]$	7.3	14.6	29.2	36.5
$\mathbf{E}[Penalty\ time]$	90.97 hours	179.82 hours	357.82 hours	450.15 hours

$\bar{P}_E$	$605.071(10^3\$)$	$572.705(10^3\$)$	$507.965(10^3\$)$	$475.496(10^3\$)$
$\bar{P}_{MC}$	$75.387(10^3\$)$	$113.084(10^3\$)$	$188.504(10^3\$)$	$226.514(10^3\$)$
$E[\Pi_M(P_E)]$	$298.271(10^3\$)$	$259.105(10^3\$)$	$180.765(10^3\$)$	$141.496(10^3\$)$
$E[\Pi_A(P_{MC})]$	$33.429(10^3\$)$	$29.295(10^3\$)$	$21.035(10^3\$)$	$17.005(10^3\$)$

Source: The author

Table 10 shows that $E[Penalty\ Time]$, $E[N(L_1)]$, $E[N(L-L_1)]$, and $\bar{P}_{MC}$ increase and $\bar{P}_E$ decrease as λ increases. Furthermore, when the equipment's reliability decreases (λ increases), the manufacturer and the agent's maximum expected profit also reduces. As a result, when the reliability is low, the players' payoff decreases since their costs increase.

We mention that $\bar{P}_{MC}$ increases as λ increases since this price includes the cost of the penalty time that also increases with λ.

3.9 Concluding and remarks

This paper showed a three-level service contract defined under a hierarchical structure for a manufacturer, an agent, and a consumer. The provision of the base warranty represents a way of convincing the consumer to buy the product, and maintenance services carried out by the agent after the base warranty period aim the consumer keeps the item along with its lifetime.

The model emphasized how the manufacturer and the agent can define pricing to extract the consumer surplus through a coalition. The equilibrium prices related to the sale of the product (including the base warranty costs) and maintenance services are determined via the Shapley value that provides a unique solution, emphasizing the marginal contribution of each decision-maker. The summation of the equilibrium prices is equal to the consumer surplus.

The game theory approach was applied due to the presence of many participants with different goals and strategies. The cooperative game establishes a coalition between the manufacturer and the agent that competes against the consumer (non-cooperative approach). For a given price structure (P_E, P_{MC}), the consumer replies, buying or not buying the product with maintenance services. As a result, the non-cooperative approach is via a sequential game whose solution is via the subgame-perfect Nash equilibrium.

There is much scope extending the present work. For example, the manufacturer and the agent deal with only one warranty option, and they price it. We could add other warranty options with different coverage mechanisms and decision variables - such as L_1. In addition, we could consider the risk parameter to model the players' preferences as seen in (ASGHARIZADEH; MURTHY, 2000; ESMAEILI; SHAMSI GAMCHI; ASGHARIZADEH, 2014; MURTHY; ASGHARIZADEH, 1998, 1999).

Finally, the decision problem could be modeled considering two maintenance agents, and the consumer chooses one of them to carry out the maintenance service after the base warranty period. Thus, the model would have four players: a manufacturer, two maintenance agents, and a consumer.

"*Persistence and determination alone are omnipotent*."
Calvin Coolidge

4 CONCLUSIONS

In this thesis, we have shown how different relationships among three decision-makers can affect the decisions they could make, the payoffs they could gain, and equilibrium. We focus on the pricing strategies for warranty policies from a game theory perspective. Two theoretical models formed this document. Furthermore, we have determined the conditions under which each pricing strategy is the most profitable.

The first model is applied to the maintenance outsourcing context, defined between the consumer and the agent. We use subgame-perfect Nash equilibrium to find the equilibrium strategies for each decision-maker. The model presents the following situation, the product owner has a product (repairable system) and outsources maintenance actions to the agent who shows four different types of warranty. Hence, the agent prices warranty options, and the consumer chooses one of them. Our results illustrate that under complete information, the agent can extract all consumer surplus. In this context, all warranty options provide the same expected gain to the consumer, zero. Besides, the model presented the condition when the consumer does not know the correct value of λ can obtain a positive profit since the agent does not price the consumer's reservation prices correctly.

The second model extends to the first one by including a new decision-maker, the manufacturer. As a result, we presented a three-person game with a hierarchical structure with double subordination. For this model, the manufacturer prices the product (including the base warranty costs), and the agent defines maintenance services that starts after the base warranty period. Through these two prices, the consumer decides whether to buy the product with maintenance services. We assume that the consumer does not abandon the item after the base warranty period. Equilibrium prices are given via a mix of cooperative and non-cooperative solutions. The model brings a coalition between the manufacturer and the agent that extract the consumer surplus. The equilibrium prices and payoffs are defined via the Shapley value. Note that the non-cooperative approach is between the coalition manufacturer-agent and the consumer via a sequential game. For each pair of prices, the consumer replies. The non-cooperative solution is via the subgame-perfect Nash equilibrium.

For both models developed, we apply the computer simulation to estimate the expected value of the warranty costs for some warranty options associated with a repairable system. This thesis also emphasized that simulation can be practical to estimate some reliability-related performance measures.

The main limitations of this work can be seen from two perspectives. Firstly, the failure-repair-failure cycle of the product. Both models followed the homogeneous Poisson process. Another stochastic process widely used in reliability modeling is the non-homogeneous Poisson process (BLISCHKE; MURTHY, 1994). Thus, we could analyze the impact of players' payoff and equilibrium strategies for this new context. The second limitation is related to payoff functions. We only consider profit functions to compute the equilibrium. A more interesting situation is to apply a utility function with a risk parameter to define the preference of the decision-makers (MURTHY; JACK, 2014).

In our future works, we would like to apply our framework for concrete maintenance service contracts, analyzing prices jointly with the design of the warranty policy. Furthermore, we can use estimation techniques for examining the failure data along with the coverage period. Another relevant research direction is to design warranty options with two dimensions. In our models, we consider the lifetime as the coverage. We could use another variable, such as the product’s usage, to define warranty policies.

This study represents the completion of a cycle. I thank those who helped me, and I wish my grandmother were alive.

"*The fear of the LORD is the beginning of knowledge; fools despise wisdom and instruction*."
Proverbs 1:7

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